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# Three dimensional vortices in the nonlinear wave equation

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## Abstract

We prove the existence of rotating solitary waves (vortices) for the nonlinear Klein-Gordon equation with nonnegative potential, by finding nonnegative cylindrical solutions to the standing equation

$$-\Delta u + \frac{\mu}{|y|^2}u + \lambda u = g(u), \quad u \in H^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} \frac{u^2}{|y|^2} dx < \infty, \quad (\dagger)$$

where  $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ ,  $N > k \geq 2$ ,  $\mu > 0$  and  $\lambda \geq 0$ . The nonnegativity of the potential makes the equation suitable for physical models and guarantees the well-posedness of the corresponding Cauchy problem, but it prevents the use of standard arguments in providing the functional associated to  $(\dagger)$  with bounded Palais-Smale sequences.

## 1 Introduction and main results

In this paper we are concerned with the existence of *vortices* for the nonlinear wave equation

$$\square \psi + W'(\psi) = 0, \quad (1)$$

where  $W'$  is (under the standard identification between  $\mathbb{C}$  and  $\mathbb{R}^2$ ) the gradient of a  $C^1$  potential function  $W : \mathbb{C} \rightarrow \mathbb{R}$  satisfying  $W(e^{i\theta}\psi) = W(\psi)$ , that is,

$$W(\psi) = V(|\psi|) \quad \text{and} \quad W'(\psi) = V'(|\psi|) \frac{\psi}{|\psi|} \quad \text{for some } V \in C^1(\mathbb{R}; \mathbb{R}). \quad (2)$$

Roughly speaking, a *vortex* is a solitary wave with nonvanishing angular momentum. A *solitary wave* is a nonsingular solution which travels as a localized

packet in such a way that the energy is conserved in time in the region of space occupied by the wave. A solitary wave bears not only the *energy*

$$\mathcal{E}(\psi) = \int_{\mathbb{R}^3} \left[ \frac{1}{2} |\partial_t \psi|^2 + \frac{1}{2} |\nabla \psi|^2 + W(\psi) \right] dx \quad (3)$$

but also the other integrals of the motion, such as the *angular momentum*

$$\mathbf{M}(\psi) = \text{Re} \int_{\mathbb{R}^3} \overline{\partial_t \psi} (\mathbf{x} \times \nabla \psi) dx, \quad (4)$$

which represent intrinsic properties of particles. In addition, the solitary waves of (1) exhibit all the most characteristic features of relativistic particles, such as the equivalence between mass and energy. Owing to this particle-like behaviour, solitary waves can thus be regarded as a model for extended particles, in contrast with point particles, and they arise in many problems of mathematical physics, such as classical and quantum field theory, nonlinear optics, fluid mechanics, plasma physics and cosmology (see, for instance, [36], [26], [22]). For an introduction to the theory of solitary waves in nonlinear field equations we refer, e.g., to [3], [9], [32].

Here we are interested in the existence of vortices of equation (1) with non-negative potentials, that is,

$$W \geq 0 \quad \text{and} \quad \mathbf{M}(\psi) \neq 0.$$

Observe that  $W \geq 0$  implies  $\mathcal{E} \geq 0$ , which is an important request for the consistence of physical models related to the equation since the existence of field configurations with negative energy would yield negative masses. Furthermore, the positivity of the energy also provides good *a priori* estimates for the solutions of the corresponding Cauchy problem and these estimates allow to prove that, under very general assumptions on  $W$ , the problem is well posed (cf. [9]).

The most natural way for finding solitary waves for (1) is to look for *static waves*, i.e., time-independent solutions of the form

$$\psi(t, x) = \psi_0(x),$$

and then to obtain travelling waves by Lorentz transforming. Unfortunately, this forces to assume that  $W$  takes negative values, for it is well known, since the renowned paper [19] of Derrick, that  $W \geq 0$  implies that any finite-energy static solution of (1) is necessarily trivial.

Such a difficulty can be overcome by looking for *standing waves*, namely, finite-energy solutions having the following form:

$$\psi(t, x) = \psi_0(x) e^{-i\omega_0 t}, \quad \omega_0 > 0. \quad (5)$$

In the literature a lot of work has been done in proving the existence of standing waves in the case in which  $\psi_0(x) \in \mathbb{R}$  (we recall, e.g., [12], [13], [29], [30], [31]). Also in the physical literature there are many papers dealing with this topic,

among which we recall the pioneering paper of Rosen [27] and the first rigorous existence paper [16]. In physics, the spherically symmetric standing waves have been called *Q-balls* by Coleman in [15] and this is the name used in all the subsequent papers.

From the results of [12] (see also [9]) it follows that, if  $W$  satisfies (2) together with

- (i)  $V \geq 0$  and  $V(0) = 0$
- (ii)  $V'(u) = \Omega^2 u + O(u^{q-1})$  as  $u \rightarrow 0^+$  for some  $\Omega^2 > 0$  and  $q > 2$
- (iii)  $V(u_0) < \frac{1}{2}\Omega^2 u_0^2$  for some  $u_0 > 0$ ,

then, setting

$$\Omega_0 := \inf \{ \omega > 0 : V(u) < \frac{1}{2}\omega^2 u^2 \text{ for some } u > 0 \}, \quad (6)$$

equation (1) has standing waves (5) with  $\psi_0(x) \in \mathbb{R}$  for every frequency  $\omega_0 \in (\Omega_0, \Omega)$ , where the limit value  $\omega_0 = \Omega$  is also admitted if  $q > 6$  in (ii) (actually, for  $\omega_0 \in (\Omega_0, \Omega)$  the result holds also replacing (ii) with  $V''(0) = \Omega^2 > 0$ ).

However  $\psi_0(x) \in \mathbb{R}$  implies  $\mathbf{M}(\psi) = 0$  and so, in order to get vortices, one has to consider complex valued  $\psi_0$ 's.

Making an ansatz of the form

$$\psi(t, x) = u(x) e^{i(k_0 \theta(x) - \omega_0 t)}, \quad u(x) \geq 0, \quad \theta(x) \in \mathbb{R}/2\pi\mathbb{Z}, \quad \omega_0 > 0, k_0 \neq 0, \quad (7)$$

equation (1) is equivalent to the system

$$\begin{cases} -\Delta u + k_0^2 |\nabla \theta|^2 u - \omega_0^2 u + V'(u) = 0 \\ u \Delta \theta + 2 \nabla u \cdot \nabla \theta = 0. \end{cases}$$

Moreover, if we denote  $x = (y, z) = (y_1, y_2, z)$ , assume  $u(y, z) = u(|y|, z)$  and choose the angular coordinate in  $\mathbb{R}^3$  as phase function, that is,

$$\theta(x) := \begin{cases} \arctan(y_2/y_1) & \text{if } y_1 > 0 \\ \arctan(y_2/y_1) + \pi & \text{if } y_1 < 0 \\ \pi/2 & \text{if } y_1 = 0 \text{ and } y_2 > 0 \\ -\pi/2 & \text{if } y_1 = 0 \text{ and } y_2 < 0, \end{cases} \quad (8)$$

we get

$$\Delta \theta = 0, \quad \nabla \theta \cdot \nabla u = 0, \quad |\nabla \theta|^2 = \frac{1}{|y|^2},$$

so that the above system reduces to

$$-\Delta u + \frac{k_0^2}{|y|^2} u + V'(u) = \omega_0^2 u \quad \text{in } \mathbb{R}^3 \quad (9)$$

and direct computations show that (3) and (4) become

$$\mathcal{E}\left(u(x)e^{i(k_0\theta(x)-\omega_0 t)}\right) = \int_{\mathbb{R}^3} \left[ \frac{1}{2} |\nabla u|^2 + \frac{1}{2} \left( \frac{k_0^2}{|y|^2} + \omega_0^2 \right) u^2 + V(u) \right] dx \quad (10)$$

$$\mathbf{M}\left(u(x)e^{i(k_0\theta(x)-\omega_0 t)}\right) = \left(0, 0, -\omega_0 k_0 \int_{\mathbb{R}^3} u^2 dx\right). \quad (11)$$

By studying equation (9) we will prove the following result.

**Theorem 1** *Let  $W : \mathbb{C} \rightarrow \mathbb{R}$  satisfy (2) and assume conditions (i), (ii), (iii). Then equation (1) has nonzero finite-energy classical solutions of the form (7)-(8) for every  $k_0 \neq 0$  and  $\omega_0 \in (\Omega_0, \Omega)$ , where  $\Omega_0$  is given by (6) and the limit value  $\omega_0 = \Omega$  is also admitted if  $q > 6$ .*

Notice that  $\Omega_0 < \Omega$  by assumption (iii), so that the interval  $(\Omega_0, \Omega)$  is nonempty. The finite energy and angular momentum of the solutions we find are given by (10) and (11), and the angular momentum does not vanish since  $u$  is nonzero.

We observe that the assumptions of Theorem 1 are satisfied for example by the model potential

$$W(\psi) = \frac{1}{2} \Omega^2 |\psi|^2 - \frac{b}{q} |\psi|^q + \frac{1}{p} |\psi|^p, \quad \Omega \neq 0, \quad p > q > 2,$$

which is nonnegative provided that  $b > 0$  is small enough.

In the physical literature, the existence of solitary waves with nonvanishing angular momentum in classical field theory seems to be an interesting open issue, which has been recently addressed in a number of publications (see for instance [33], [17], [14] and the references therein). In particular, the existence of vortices for equation (1) has been investigated in [21] and [34], for very particular potentials.

From the mathematical viewpoint, the existence of vortices has been studied in [11] and [4] (see also [7], [8], [10], [18] for related equations and results), but the requirement  $W \geq 0$  was not permitted by the results there. We also mention a forthcoming paper [5], where the problem of vortices with prescribed charge is investigated.

**Remark 2** *Theorem 1 also gives travelling solitary waves with nonvanishing angular momentum, since, by Lorentz invariance, a solution  $\psi_{\mathbf{v}}$  travelling with any vector velocity  $\mathbf{v}$  can be obtained from a standing one by boosting. For instance, if  $\psi(t, x) = u(x)e^{i(k_0\theta(x)-\omega_0 t)}$  is a standing solution and  $\mathbf{v} = (0, 0, v)$ ,  $|v| < 1$ , then*

$$\psi_{\mathbf{v}}(t, x) = u(y, \gamma(z - vt)) e^{i(k_0\theta(x) - \omega_0\gamma(t - vz))}, \quad \gamma = (1 - v^2)^{-1/2},$$

*is a solution representing a bump which travels in the  $z$ -direction with speed  $v$ .*

**Remark 3** *The same arguments leading to Theorem 1 also yield the existence of standing and travelling rotating solitary waves for the nonlinear Schrödinger equation*

$$i\partial_t \psi = -\Delta \psi + W'(\psi), \quad \psi(t, x) \in \mathbb{C}, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3. \quad (12)$$

Actually we stated the result for the nonlinear Klein-Gordon equation (1) because it is for this equation that, as already mentioned, the assumption  $W \geq 0$  has special importance on physical grounds.

According to the previous discussion, the proof of Theorem 1 relies on finding nonnegative symmetric solutions to equation (9) with suitable integrability properties. In fact we will perform this study in a more general situation, that is, we will study the existence of nontrivial solutions to the following problem:

$$\begin{cases} -\Delta u + \frac{\mu}{|y|^2} u + \lambda u = g(u) & \text{in } \mathbb{R}^N \\ u(y, z) = u(|y|, z) \geq 0 & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} \frac{u^2}{|y|^2} dx < \infty \end{cases} \quad (13)$$

where  $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$  with  $N > k \geq 2$ , the nonlinearity  $g : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and such that  $g(0) = 0$ , and  $\mu > 0$  and  $\lambda \geq 0$  are real constants. More precisely, we introduce the spaces

$$H := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} \frac{u^2}{|y|^2} dx < \infty \right\}, \quad H_s := \{ u \in H : u(y, z) = u(|y|, z) \} \quad (14)$$

and look for weak solutions in the sense of the following definition: we name *weak solution* to problem (13) any nonnegative  $u \in H_s$  such that

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla h \, dx + \mu \int_{\mathbb{R}^N} \frac{uh}{|y|^2} dx + \lambda \int_{\mathbb{R}^N} uh \, dx = \int_{\mathbb{R}^N} g(u) h \, dx \quad \text{for all } h \in H. \quad (15)$$

Regarding the nonlinearity, we will assume

$$(g_0) \quad \int_0^{t_0} g(s) \, ds > \lambda t_0^2 / 2 \text{ for some } t_0 > 0$$

$$(g_1) \quad g(t) = O(t^{q-1}) \text{ as } t \rightarrow 0^+ \text{ for some } q > 2$$

together with one of the following conditions:

$$(g_2) \quad g(\beta) = 0 \text{ for some } \beta > \beta_0 := \inf\{t > 0 : \int_0^t g(s) \, ds > \lambda t^2 / 2\}$$

$$(g_3) \quad g(t) = O(t^{p-1}) \text{ as } t \rightarrow +\infty \text{ for some } p \in (1, 2^*)$$

where  $2^* := 2N/(N-2)$  denotes the critical exponent of Sobolev embedding.

The relationship between (9) and (13) is clear: writing

$$V(|\psi|) = \frac{1}{2}\Omega^2 |\psi|^2 - G(|\psi|),$$

equation (9) reduces to the equation of (13) with  $\lambda = \Omega^2 - \omega_0^2$  and  $g = G'$ . This leads to not assuming the well known superquadraticity condition due to Ambrosetti and Rabinowitz [1], namely

$$\sigma G(t) \leq G'(t)t \quad \text{for some } \sigma > 2 \text{ and all } t \in \mathbb{R}, \quad (16)$$

since, together with  $(\mathbf{g}_0)$ , it implies

$$G(|\psi|) \geq (\text{const.}) |\psi|^\sigma \quad \text{for } |\psi| \text{ large}$$

and thus forces  $W$  to take negative values.

Our existence result is the following.

**Theorem 4** *Let  $N > k \geq 2$ ,  $\mu > 0$  and  $\lambda \geq 0$ . Assume that  $g \in C(\mathbb{R}; \mathbb{R})$  satisfies  $(\mathbf{g}_0)$ ,  $(\mathbf{g}_1)$  and at least one of hypotheses  $(\mathbf{g}_2)$  and  $(\mathbf{g}_3)$ , with  $q > 2^*$  if  $\lambda = 0$  and  $p > 2$  if  $\lambda > 0$ . Then problem (13) has at least a nonzero weak solution, which satisfies  $\|u\|_{L^\infty(\mathbb{R}^N)} \leq \beta$  if  $(\mathbf{g}_2)$  holds.*

The proof of Theorem 4 will be given in Section 4, where a solution to (13) will be found as a mountain-pass critical point of the Euler functional associated to the equation. The difficulty of obtaining a bounded Palais-Smale sequence without the aid of condition (16) will be preliminarily tackled in Section 3.

As a matter of fact, the case  $\lambda > 0$  can also be studied by suitably adapting the constrained minimization technique of [12], but such an argument fails for  $\lambda = 0$ , when the  $H^1$  variational theory does not apply (in particular one cannot obtain compactness by exploiting well known results such as [35, Lemma 1.21]) and a different approach is needed.

Still concerning the case  $\lambda = 0$ , we also observe that Theorem 4 actually gives a version of the results of [4] without (16) and that a similar result was announced in [24] without proof.

Finally, we remark that Theorem 4 applies to more general situations than the ones needed to deduce Theorem 1. For instance it also admits pure power nonlinearities, or, more generally, nonlinearities which may satisfy Ambrosetti-Rabinowitz condition.

We conclude this introductory section by collecting the notations of most frequent use throughout the paper.

- Given  $N, k \in \mathbb{N}$ ,  $N > k \geq 2$ , we shall always write  $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ .
- $O(k)$  is the orthogonal group of  $\mathbb{R}^k$ .
- By  $u(y, z) = u(|y|, z)$  we always mean  $u(y, z) = u(Ry, z)$  for all  $R \in O(k)$  and almost every  $(y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ .
- For any  $r \in \mathbb{R}$  we set  $r_+ := (|r| + r)/2$  and  $r_- := (|r| - r)/2$ , so that  $r = r_+ - r_-$  with  $r_+, r_- \geq 0$ .



- $|A|$  and  $\chi_A$  respectively denote the  $d$ -dimensional Lebesgue measure and the characteristic function of any measurable set  $A \subseteq \mathbb{R}^d$ ,  $d \geq 1$ .
- By  $\rightarrow$  and  $\rightharpoonup$  we respectively mean *strong* and *weak* convergence in a Banach space  $E$ , whose dual space is denoted by  $E'$ . The open ball  $B_r(u_0) := \{u \in E : \|u - u_0\|_E < r\}$  shall be simply denoted by  $B_r$  when  $E = \mathbb{R}^N$  and  $u_0 = 0$ .
- $\hookrightarrow$  denotes *continuous* embeddings.
- $C_c^\infty(A)$  is the space of the infinitely differentiable (real or complex) functions with compact support in the open set  $A \subseteq \mathbb{R}^d$ ,  $d \geq 1$ .
- If  $1 \leq p \leq \infty$  then  $L^p(A)$  and  $L_{loc}^p(A)$  are the usual Lebesgue spaces (for any measurable set  $A \subseteq \mathbb{R}^d$ ,  $d \geq 1$ ). We recall in particular that  $u_n \rightarrow 0$  in  $L_{loc}^p(\mathbb{R}^d)$  if and only if  $u_n \rightarrow 0$  in  $L^p(B_r)$  for every  $r > 0$ .
- $2^* := 2N/(N-2)$ ,  $N \geq 3$ , is the critical exponent for the Sobolev embedding.
- $H^1(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$  and  $D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$  are the usual Sobolev spaces.

## 2 Preliminaries

In this section we study the functional framework in which problem (13) can be cast into a variational formulation. In particular, Subsection 2.1 is devoted to a brief description of some weighted Sobolev spaces naturally related to problem (13), while in Subsection 2.2 we derive a variational principle for recovering weak solutions of problem (13) as critical points of a suitable functional (Proposition 7), of which we also give some relevant properties (Lemmas 8 and 9).

Throughout the section we assume  $N > k \geq 2$ ,  $\mu > 0$  and  $\lambda \geq 0$ .

### 2.1 Weighted Sobolev spaces

In order to emphasize the role of  $\lambda$ , for  $\lambda > 0$  we respectively denote by  $H_\lambda$  and  $H_{\lambda,s}$  the Hilbert spaces  $H$  and  $H_s$  of (14) endowed with the norm defined by

$$\|u\|_\lambda^2 := \int_{\mathbb{R}^N} |\nabla u|^2 dx + \mu \int_{\mathbb{R}^N} \frac{u^2}{|y|^2} dx + \int_{\mathbb{R}^N} \lambda u^2 dx \quad \text{for all } u \in H_\lambda, \quad (17)$$

which is induced by the inner product

$$(u | v)_\lambda := \int_{\mathbb{R}^N} \nabla u \cdot \nabla v dx + \mu \int_{\mathbb{R}^N} \frac{uv}{|y|^2} dx + \int_{\mathbb{R}^N} \lambda uv dx \quad \text{for all } u, v \in H_\lambda. \quad (18)$$

Clearly  $H_{\lambda,s} \hookrightarrow H_\lambda \hookrightarrow H^1(\mathbb{R}^N)$  and, by well known embeddings of  $H^1(\mathbb{R}^N)$ , one has that  $H_\lambda \hookrightarrow L^p(\mathbb{R}^N)$  for  $2 \leq p \leq 2^*$  and  $H_\lambda \hookrightarrow L_{loc}^p(\mathbb{R}^N)$  for  $1 \leq p \leq 2^*$ . In particular, the latter embedding is compact if  $p < 2^*$  and thus it assures that weak convergence in  $H_\lambda$  implies, up to a subsequence, almost everywhere convergence in  $\mathbb{R}^N$ .

If  $\lambda = 0$ , the natural functional spaces associated to equation (13) are instead

$$H_0 := \left\{ u \in D^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \frac{u^2}{|y|^2} dx < \infty \right\}$$

$$H_{0,s} := \{ u \in H_0 : u(y, z) = u(|y|, z) \}$$

equipped with the norm and inner product still given by (17)-(18). Clearly  $H_\lambda = H_0 \cap L^2(\mathbb{R}^N) \hookrightarrow H_0 \hookrightarrow D^{1,2}(\mathbb{R}^N)$  and  $H_{\lambda,s} = H_{0,s} \cap L^2(\mathbb{R}^N) \hookrightarrow H_{0,s} \hookrightarrow H_0$  for any  $\lambda > 0$ . Moreover, by well known embeddings of  $D^{1,2}(\mathbb{R}^N)$ , one has  $H_0 \hookrightarrow L^{2^*}(\mathbb{R}^N)$  and  $H_0 \hookrightarrow L^p_{loc}(\mathbb{R}^N)$  with compact embedding if  $1 \leq p < 2^*$  (which also assures that weak convergence in  $H_0$  implies, up to a subsequence, almost everywhere convergence in  $\mathbb{R}^N$ ).

**Remark 5** If  $k > 2$ , from the Sobolev-Hardy inequalities [6] it follows that  $H_0 = D^{1,2}(\mathbb{R}^N)$  and the norms  $\|\cdot\|_0$  and  $\|\cdot\|_{D^{1,2}(\mathbb{R}^N)}$  are equivalent.

**Proposition 6**  $C_c^\infty((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k})$  and  $C_c^\infty((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k}) \cap H_{0,s}$  are dense in  $H_0$  and  $H_{0,s}$  respectively.

**Proof.** We divide the proof into two steps, using a standard truncation and regularization argument. Set  $\mathcal{O} := (\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k}$  for brevity and let  $X := \{u \in H_0 : \text{supp } u \text{ is compact in } \mathcal{O}\}$  and  $X_s := X \cap H_{0,s}$ .

*Step 1:*  $X$  and  $X_s$  are dense in  $H_0$  and  $H_{0,s}$ .

Fix  $\xi \in C_c^\infty(\mathbb{R})$  and  $\eta \in C^\infty(\mathbb{R})$  such that  $\xi(t) \equiv 1$  and  $\eta(t) \equiv 0$  on  $[0, 1]$ ,  $\xi(t) \equiv 0$  and  $\eta(t) \equiv 1$  on  $[2, +\infty)$ ,  $0 \leq \xi \leq 1$  and  $0 \leq \eta \leq 1$  on  $\mathbb{R}$ . For all  $n \in \mathbb{N} \setminus \{0\}$  and  $x = (y, z) \in \mathbb{R}^N$ , set  $\xi_n(x) := \xi(|x|/n)$  and  $\eta_n(x) := \eta(n^\delta |y|)$  for some  $\delta > (N-k)/k$ . Then let  $u \in H_0 \setminus \{0\}$  and define  $u_n := \xi_n \eta_n u$ , in such a way that  $\text{supp } u_n$  is compact in  $\mathcal{O}$ . Clearly  $u_n \rightarrow u$  almost everywhere on  $\mathbb{R}^N$  and  $\| |y|^{-1} (u_n - u) \|_{L^2(\mathbb{R}^N)} \rightarrow 0$  by dominated convergence. Now consider

$$\nabla u_n = \xi_n \eta_n \nabla u + u \eta_n \nabla \xi_n + u \xi_n \nabla \eta_n.$$

Again by dominated convergence one deduces that  $\xi_n \eta_n \nabla u \rightarrow \nabla u$  in  $L^2(\mathbb{R}^N)$ . On the other hand, setting  $C_1 := \max_{t \geq 0} \xi'(t)^2$ , we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} (u \eta_n)^2 |\nabla \xi_n|^2 dx &\leq \frac{1}{n^2} \int_{B_{2n} \setminus B_n} \xi'(|x|/n)^2 u^2 dx \leq \frac{C_1}{n^2} \int_{B_{2n} \setminus B_n} u^2 dx \\ &\leq \frac{C_1}{n^2} |B_{2n} \setminus B_n|^{1-2/2^*} \left( \int_{B_{2n} \setminus B_n} |u|^{2^*} dx \right)^{2/2^*} \\ &= C_1 |B_1|^{2/N} (2^N - 1)^{2/N} \left( \int_{B_{2n} \setminus B_n} |u|^{2^*} dx \right)^{2/2^*} \end{aligned}$$

where the last integral goes to zero as  $n \rightarrow \infty$  because  $u \in L^{2^*}(\mathbb{R}^N)$ . Finally, setting  $A_n := \{x \in \mathbb{R}^N : 1 < |n^\delta y| < 2, |z| < 2n\}$  and  $C_2 := \max_{t \geq 0} \eta'(t)^2$ , we

have

$$\begin{aligned} \int_{\mathbb{R}^N} u^2 \xi_n^2 |\nabla \eta_n|^2 dx &= \int_{A_n} u^2 \xi_n^2 |\nabla \eta_n|^2 dx \leq n^{2\delta} \int_{A_n} \eta' (n^\delta |y|)^2 u^2 dx \\ &\leq C_2 n^{2\delta} \int_{A_n} |y|^2 \frac{u^2}{|y|^2} dx \leq 4C_2 \int_{A_n} \frac{u^2}{|y|^2} dx \end{aligned}$$

where the last integral goes to zero as  $n \rightarrow \infty$  because  $|y|^{-2} u^2 \in L^1(\mathbb{R}^N)$  and  $|A_n| = Cn^{N-k-\delta k} = o(1)_{n \rightarrow \infty}$  for some constant  $C > 0$ . Therefore  $u_n \in X$  and  $u_n \rightarrow u$  in  $H_0$ . Since  $u_n$  only depends on  $|y|$  if  $u \in H_{0,s}$ , the claim is proved.

*Step 2:*  $C_c^\infty(\mathcal{O})$  and  $C_c^\infty(\mathcal{O}) \cap X_s$  are dense in  $X$  and  $X_s$  (with respect to  $\|\cdot\|_0$ ).

Fix any  $u \in X$ ,  $u \neq 0$ , and let  $0 < r_0 < r$  be such that  $\text{supp } u \subset A := \{x \in \mathbb{R}^N : r_0 < |y| < r, |z| < r\}$ . Define

$$u_\varepsilon(x) := \int_A u(x') \rho_\varepsilon(x - x') dx' \quad \text{for all } x \in \mathbb{R}^N \text{ and } \varepsilon \in \left(0, \frac{r_0}{2}\right)$$

where  $\{\rho_\varepsilon\} \subset C_c^\infty(\mathbb{R}^N)$  is a family of radial mollifiers, that is,  $\rho_\varepsilon \geq 0$ ,  $\text{supp } \rho_\varepsilon \subset B_\varepsilon$  and  $\|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} = 1$ . By standard arguments,  $u_\varepsilon \in C_c^\infty(\mathbb{R}^N)$  and  $u_\varepsilon \rightarrow u$  in  $D^{1,2}(\mathbb{R}^N)$  and  $L_{loc}^2(\mathbb{R}^N)$  as  $\varepsilon \rightarrow 0$ . Moreover  $\varepsilon < r_0/2$  implies that both  $\text{supp } u$  and  $\text{supp } u_\varepsilon$  lie in  $K := \{x \in \mathbb{R}^N : r_0/2 \leq |y| \leq 2r, |z| \leq 2r\}$ , whence one deduces

$$\int_{\mathbb{R}^N} \frac{(u - u_\varepsilon)^2}{|y|^2} dx = \int_K \frac{(u - u_\varepsilon)^2}{|y|^2} dx \leq \frac{4}{r_0^2} \int_K (u - u_\varepsilon)^2 dx = o(1)_{\varepsilon \rightarrow 0}.$$

Therefore  $u_\varepsilon \in C_c^\infty(\mathcal{O})$  and  $u_\varepsilon \rightarrow u$  in  $H_0$  as  $\varepsilon \rightarrow 0$ . Since one easily checks that  $u \in X_s$  implies  $u_\varepsilon(Ry, z) = u_\varepsilon(y, z)$  for all  $R \in O(k)$  and almost every  $(y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ , the proof is complete. ■

## 2.2 Variational approach

Let  $g \in C(\mathbb{R}; \mathbb{R})$  satisfy the hypotheses of Theorem 4. Set  $\chi := \chi_{(0,\beta)}$  if **(g<sub>2</sub>)** holds,  $\chi := \chi_{(0,+\infty)}$  otherwise. Then define

$$f(t) := \chi(t) g(t) \quad \text{and} \quad F(t) := \int_0^t f(s) ds \quad \text{for all } t \in \mathbb{R}. \quad (19)$$

So, in any case, from **(g<sub>0</sub>)** one deduces that

$$\textbf{(F}_0\textbf{)} \quad \exists t_0 > 0 \text{ such that } F(t_0) > \lambda t_0^2/2.$$

Moreover, if  $\lambda > 0$ , it is not restrictive to assume  $q < 2^*$  in **(g<sub>1</sub>)** and the hypotheses of Theorem 4 imply

$$\textbf{(f}_V\textbf{)} \quad \exists m > 0, \forall t \in \mathbb{R}, |f(t)| \leq m \max\{|t|^{p-1}, |t|^{q-1}\} \quad (\text{where } p, q \in (2, 2^*))$$

(**F<sub>∨</sub>**)  $\exists M > 0, \forall t \in \mathbb{R}, |F(t)| \leq M \max\{|t|^p, |t|^q\}$  (where  $p, q \in (2, 2^*)$ )

whereas, if  $\lambda = 0$ , one deduces

(**f<sub>∧</sub>**)  $\exists m > 0, \forall t \in \mathbb{R}, |f(t)| \leq m \min\{|t|^{p-1}, |t|^{q-1}\}$  (where  $1 < p < 2^* < q$ )

which yields in particular

(**f<sub>\*</sub>**)  $|f(t)| \leq m |t|^{2^*-1}$  for all  $t \in \mathbb{R}$

(**F<sub>\*</sub>**)  $\exists M > 0$  such that  $|F(t)| \leq M |t|^{2^*}$  for all  $t \in \mathbb{R}$ .

Thanks to (**f<sub>∨</sub>**), (**F<sub>∨</sub>**), (**f<sub>\*</sub>**), (**F<sub>\*</sub>**) and the continuous embeddings  $H_\lambda \hookrightarrow L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$  for  $\lambda > 0$  and  $H_0 \hookrightarrow L^{2^*}(\mathbb{R}^N)$ , one checks (see for example [23]) that the functional  $I_\lambda : H_\lambda \rightarrow \mathbb{R}$  defined (for any  $\lambda \geq 0$ ) by

$$I_\lambda(u) := \frac{1}{2} \|u\|_\lambda^2 - \int_{\mathbb{R}^N} F(u) dx \quad \text{for all } u \in H_\lambda \quad (20)$$

is of class  $C^1$  on  $H_\lambda$  and has Fréchet derivative  $I'_\lambda(u) \in H'_\lambda$  at any  $u \in H_\lambda$  given by

$$I'_\lambda(u)h = (u | h)_\lambda - \int_{\mathbb{R}^N} f(u)h dx \quad \text{for all } h \in H_\lambda. \quad (21)$$

We now show that the set of weak solutions to problem (13) equals the set of critical points of the functional

$$J_\lambda := I_{\lambda|_{H_{\lambda,s}}} : H_{\lambda,s} \rightarrow \mathbb{R}$$

defined as the restriction of  $I_\lambda$  to  $H_{\lambda,s}$ , which is obviously such that  $J_\lambda \in C^1(H_{\lambda,s}; \mathbb{R})$  and  $J'_\lambda(u)h = I'_\lambda(u)h$  for all  $u, h \in H_{\lambda,s}$ . Observe that weak solutions belong to  $H^1(\mathbb{R}^N)$  by definition, while  $H_0 \not\subseteq H^1(\mathbb{R}^N)$  (cf. Remark 5).

**Proposition 7** *Every critical point of  $J_\lambda$  is a weak solution to problem (13) and, if (g<sub>2</sub>) holds, it satisfies  $u \leq \beta$  almost everywhere in  $\mathbb{R}^N$ .*

**Proof.** Let  $u \in H_{\lambda,s}$  be such that  $J'_\lambda(u)h = 0$  for all  $h \in H_{\lambda,s}$ . Then, by virtue of the principle of symmetric criticality [25],  $u$  is a critical point of  $I_\lambda$ , i.e.,  $I'_\lambda(u)h = 0$  for all  $h \in H_\lambda$ . Now, using  $h = u_- \in H_{\lambda,s}$  as test function in (21), one obtains  $\|u_-\|_\lambda = 0$ , that is,  $u \geq 0$ . If  $f = \chi_{(0,+\infty)}g$ , this implies  $f(u) = g(u)$  and thus (15) holds by (21). Otherwise, if  $f = \chi_{(0,\beta)}g$ , we compute (21) for  $h = (u - \beta)_+ \in H_{\lambda,s}$  and, since  $f(u)(u - \beta)_+$  vanishes almost everywhere in  $\mathbb{R}^N$ , we get

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} \nabla u \cdot \nabla (u - \beta)_+ dx + \mu \int_{\mathbb{R}^N} \frac{u(u - \beta)_+}{|y|^2} dx + \int_{\mathbb{R}^N} \lambda u (u - \beta)_+ dx \\ &\geq \int_{\mathbb{R}^N} \nabla u \cdot \nabla (u - \beta)_+ dx = \int_{\mathbb{R}^N} |\nabla (u - \beta)_+|^2 dx. \end{aligned}$$

This implies  $(u - \beta)_+ = 0$ , i.e.,  $u \leq \beta$ , which yields  $f(u) = g(u)$  and thus proves (15) again. Finally, one deduces that  $u \in H^1(\mathbb{R}^N)$  also if  $\lambda = 0$  thanks to [4, Proposition 6]. ■

The next lemma assures that weak limits of criticizing sequences are actually critical points for  $J_\lambda$ .

**Lemma 8** *For any  $h \in H_{\lambda,s}$  the mapping  $J'_\lambda(\cdot)h : H_{\lambda,s} \rightarrow \mathbb{R}$  is sequentially weakly continuous.*

**Proof.** We assume  $\lambda > 0$  and follow the argument of [4, Proposition 14], where the claim of the lemma has already been proved for  $\lambda = 0$ . Of course we need only consider the nonlinear term of the mapping, so fix  $h \in H^1(\mathbb{R}^N)$  and show the sequential weak continuity on  $H^1(\mathbb{R}^N)$  of the mapping  $u \mapsto \int_{\mathbb{R}^N} f(u) h \, dx$ . Accordingly, assume  $u_n \rightharpoonup u$  in  $H^1(\mathbb{R}^N)$  and, with a view to arguing by density, let  $\varphi \in C_c^\infty(\mathbb{R}^N)$  and let  $r > 0$  be such that  $\text{supp } \varphi \subset B_r$ . Since  $u_n \rightarrow u$  in  $L^{p-1}(B_r) \cap L^{q-1}(B_r)$  and condition  $(\mathbf{f}_\vee)$  assures the continuity of the Nemytskiĭ operator  $f : L^{p-1}(B_r) \cap L^{q-1}(B_r) \rightarrow L^1(B_r)$ , one readily has  $\int_{\mathbb{R}^N} |f(u_n) - f(u)| |\varphi| \, dx = o(1)$  as  $n \rightarrow \infty$ . Then, by the boundedness of  $\{u_n\}$  in  $H^1(\mathbb{R}^N)$ , there exists a constant  $C > 0$  (independent from  $\varphi$  and  $n$ ) such that

$$\begin{aligned} & \int_{\mathbb{R}^N} |f(u_n) - f(u)| |h| \, dx \\ & \leq \int_{\mathbb{R}^N} |f(u_n) - f(u)| |h - \varphi| \, dx + \int_{\mathbb{R}^N} |f(u_n) - f(u)| |\varphi| \, dx \\ & \leq \int_{\mathbb{R}^N} (|f(u_n)| + |f(u)|) |h - \varphi| \, dx + o(1)_{n \rightarrow \infty} \\ & \leq m \int_{\mathbb{R}^N} (|u_n|^{p-1} + |u_n|^{q-1} + |u|^{p-1} + |u|^{q-1}) |h - \varphi| \, dx + o(1)_{n \rightarrow \infty} \\ & \leq m \left( \|u_n\|_{L^p(\mathbb{R}^N)}^{(p-1)/p} + \|u\|_{L^p(\mathbb{R}^N)}^{(p-1)/p} \right) \|h - \varphi\|_{L^p(\mathbb{R}^N)} + \\ & \quad + m \left( \|u_n\|_{L^q(\mathbb{R}^N)}^{(q-1)/q} + \|u\|_{L^q(\mathbb{R}^N)}^{(q-1)/q} \right) \|h - \varphi\|_{L^q(\mathbb{R}^N)} + o(1)_{n \rightarrow \infty} \\ & \leq C \|h - \varphi\|_{H^1(\mathbb{R}^N)} + o(1)_{n \rightarrow \infty} \end{aligned}$$

and the density of  $C_c^\infty(\mathbb{R}^N)$  in  $H^1(\mathbb{R}^N)$  allows us to conclude. ■

We conclude this subsection with a technical lemma which emphasize the role of assumption  $(\mathbf{Fo})$  and will be useful in proving the mountain-pass geometry of  $J_\lambda$  (Lemma 10 below).

**Lemma 9** *Let  $\mathcal{A} := \{x \in \mathbb{R}^N : |y| > 1\}$ . Then there exists  $u_0 \in C_c^\infty(\mathcal{A}) \cap H_{\lambda,s}$  such that  $\int_{\mathbb{R}^N} (F(u_0) - \lambda u_0^2/2) \, dx > 0$ .*

**Proof.** Denote  $Q_{r_1, r_2} := \{x \in \mathbb{R}^N : r_1 \leq |y| \leq r_2, r_1 \leq |z| \leq r_2\}$  for  $r_2 > r_1 > 0$  and, for any  $r > 3$ , let  $\phi_r \in C_c^\infty(\mathbb{R})$  be such that

- $\phi_r(t) \equiv 1$  on  $[3, r]$
- $\phi_r(t) \equiv 0$  on  $(-\infty, 2] \cup [r+1, +\infty)$
- $0 \leq \phi_r \leq 1$  on  $\mathbb{R}$ .

Let  $t_0 > 0$  be given by **(F<sub>0</sub>)** and set  $u_r(x) := t_0 \phi_r(|y|) \phi_r(|z|)$  for all  $x \in \mathbb{R}^N$ . Clearly  $u_r \in C^\infty(\mathbb{R}^N) \cap H_{\lambda,s}$  with  $\text{supp } u_r \subseteq Q_{2,r+1}$ . Then we get

$$\begin{aligned} \int_{\mathbb{R}^N} \left( F(u_r) - \frac{\lambda}{2} u_r^2 \right) dx &= \int_{Q_{2,r+1} \setminus Q_{3,r}} \left( F(u_r) - \frac{\lambda}{2} u_r^2 \right) dx + C_0 |Q_{3,r}| \\ &\geq (C_0 + C_1) |Q_{3,r}| - C_1 |Q_{2,r+1}| = Cr^N + o(r^N) \end{aligned}$$

as  $r \rightarrow +\infty$ , where  $C_0 := F(t_0) - \lambda t_0^2/2 > 0$ ,  $C_1 := \max_{t \in [0, t_0]} |F(t) - \lambda t^2/2|$  and  $C > 0$  is a suitable constant. ■

### 3 Existence of bounded Palais-Smale sequences

Assume  $N > k \geq 2$ ,  $\mu > 0$  and  $\lambda \geq 0$ , and let  $g \in C(\mathbb{R}; \mathbb{R})$  satisfy the hypotheses of Theorem 4. The aim of this section is to prove that the functional  $J_\lambda$  defined in Subsection 2.2 as the restriction of  $I_\lambda$  to  $H_{\lambda,s}$  admits a bounded Palais-Smale sequence at its mountain-pass level  $c > 0$  (see (27) below), that is, a bounded sequence  $\{w_n\} \subset H_{\lambda,s}$  such that  $J_\lambda(w_n) \rightarrow c$  and  $J'_\lambda(w_n) \rightarrow 0$  in  $H'_{\lambda,s}$ .

In order to emphasize the different behaviour of different terms of  $J_\lambda$  in front of rescalings, we denote

$$\mathcal{F}(u) := \frac{1}{2} \|u\|_0^2 - J_\lambda(u) = \int_{\mathbb{R}^N} \left( F(u) - \frac{\lambda}{2} u^2 \right) dx \quad \text{for all } u \in H_{\lambda,s}$$

and set  $u^t := u(t^{-1} \cdot)$  for every  $u \in H_{\lambda,s}$  and  $t > 0$ . Notice that  $u^t \in H_{\lambda,s}$  with

$$\|u^t\|_0^2 = t^{-2} \int_{\mathbb{R}^N} \left( |\nabla u(t^{-1}x)|^2 + \mu \frac{u(t^{-1}x)^2}{|t^{-1}y|^2} \right) dx = t^{N-2} \|u\|_0^2 \quad (22)$$

and

$$\mathcal{F}(u^t) = \int_{\mathbb{R}^N} \left( F(u(t^{-1}x)) - \frac{\lambda}{2} u(t^{-1}x)^2 \right) dx = t^N \mathcal{F}(u). \quad (23)$$

Similarly  $\|u^t - v^t\|_\lambda^2 = t^{N-2} \|u - v\|_0^2 + t^N \|\lambda(u - v)\|_{L^2(\mathbb{R}^N)}^2$ , so that the mapping  $u \mapsto u^t$  is continuous from  $H_{\lambda,s}$  into itself.

The following lemma shows that  $J_\lambda$  has a mountain-pass geometry. Recall from Lemma 9 that we denote  $\mathcal{A} := \{x \in \mathbb{R}^N : |y| > 1\}$ .

**Lemma 10** *There exist  $\rho > 0$  and  $\bar{u} \in C_c^\infty(\mathcal{A}) \cap H_{\lambda,s}$  such that*

$$\inf_{u \in H_{\lambda,s}, \|u\|_\lambda = \rho} J_\lambda(u) > 0, \quad \|\bar{u}\|_\lambda > \rho \quad \text{and} \quad J_\lambda(\bar{u}) < 0. \quad (24)$$

**Proof.** Let  $u_0$  be the mapping of Lemma 9. Then  $u_0^t \in C_c^\infty(\mathcal{A}) \cap H_{\lambda,s}$  for all  $t > 1$  and, as  $t \rightarrow +\infty$ , one has

$$\|u_0^t\|_\lambda^2 = t^{N-2} \|u_0\|_0^2 + \lambda t^N \|u_0\|_{L^2(\mathbb{R}^N)}^2 \rightarrow +\infty \quad (25)$$

$$J_\lambda(u_0^t) = \frac{1}{2} \|u_0^t\|_0^2 - \mathcal{F}(u_0^t) = t^{N-2} \|u_0\|_0^2 - t^N \mathcal{F}(u_0) \rightarrow -\infty. \quad (26)$$

On the other hand, if  $\lambda = 0$ , by  $(\mathbf{F}_*)$  and the continuity of the embedding  $H_{0,s} \hookrightarrow L^{2^*}(\mathbb{R}^N)$  there exists  $M_0 > 0$  such that

$$J_0(u) \geq \frac{1}{2} \|u\|_0^2 - M \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \geq \frac{1}{2} \|u\|_0^2 - M_0 \|u\|_0^{2^*} \quad \text{for all } u \in H_{0,s}$$

while, if  $\lambda > 0$ , the continuous embedding  $H_{\lambda,s} \hookrightarrow L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$  together with  $(\mathbf{F}_\vee)$  assures the existence of  $M_1, M_2 > 0$  such that

$$J_\lambda(u) \geq \frac{1}{2} \|u\|_\lambda^2 - M \|u\|_{L^p(\mathbb{R}^N)}^p - M \|u\|_{L^q(\mathbb{R}^N)}^q \geq \frac{1}{2} \|u\|_\lambda^2 - M_1 \|u\|_\lambda^p - M_2 \|u\|_\lambda^q$$

for all  $u \in H_{\lambda,s}$ , where  $p, q > 2$ . This proves the first inequality of (24) for any  $\lambda \geq 0$ . By (25)-(26), we conclude by taking  $\bar{u} := u_0^t$  for  $t > 1$  large enough. ■

Hereafter we let  $\bar{u}$  be the mapping of Lemma 10.

**Lemma 11** *Let  $\{t_n\} \subset (0, +\infty)$  be a sequence such that  $t_n \rightarrow 1$ . Then  $\bar{u}^{t_n} \rightarrow \bar{u}$  in  $H_{\lambda,s}$ .*

**Proof.** First observe that  $\bar{u}^{t_n} = \bar{u}(t_n^{-1} \cdot) \rightarrow \bar{u}$  and  $\nabla \bar{u}^{t_n} = t_n^{-1} \nabla \bar{u}(t_n^{-1} \cdot) \rightarrow \nabla \bar{u}$  almost everywhere in  $\mathbb{R}^N$ , with  $\{\bar{u}^{t_n}\}$  and  $\{\nabla \bar{u}^{t_n}\}$  uniformly bounded in  $\mathbb{R}^N$ . Then let  $r > 0$  be such that  $\text{supp } \bar{u} \subset \mathcal{A} \cap B_r$  and set  $\mathcal{B} := \{x \in B_{2r} : |y| > 1/2\}$ , so that both  $\bar{u}$  and  $\bar{u}^{t_n}$  belong to  $C_c^\infty(\mathcal{B})$  for  $n$  large enough. By dominated convergence we thus conclude  $\|\bar{u}^{t_n} - \bar{u}\|_{L^2(\mathbb{R}^N)} = \|\bar{u}^{t_n} - \bar{u}\|_{L^2(\mathcal{B})} = o(1)$ ,

$$\int_{\mathbb{R}^N} \frac{(\bar{u}^{t_n} - \bar{u})^2}{|y|^2} dx = \int_{\mathcal{B}} \frac{(\bar{u}^{t_n} - \bar{u})^2}{|y|^2} dx \leq 4 \int_{\mathcal{B}} (\bar{u}^{t_n} - \bar{u})^2 dx = o(1)$$

and  $\|\nabla(\bar{u}^{t_n} - \bar{u})\|_{L^2(\mathbb{R}^N)} = \|\nabla(\bar{u}^{t_n} - \bar{u})\|_{L^2(\mathcal{B})} = o(1)$  as  $n \rightarrow \infty$ . ■

Henceforth we fix a  $\varepsilon_* > 0$  such that  $J_\lambda(u) < 0$  for all  $u \in B_{\varepsilon_*}(\bar{u})$  and, by Lemma 11, a threshold  $t_* \in (0, 1)$  such that  $\bar{u}^t \in B_{\varepsilon_*}(\bar{u})$  for all  $t \in (t_*, 1)$ .

Let us now introduce the mountain-pass level

$$c := \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} J_\lambda(u), \quad \Gamma := \{\gamma \in C([0,1]; H_{\lambda,s}) : \gamma(0) = 0, \gamma(1) = \bar{u}\}, \quad (27)$$

which is positive by Lemma 10. The existence of a Palais-Smale sequence at level  $c$  then follows from standard deformation arguments, but, as we do not assume the already mentioned Ambrosetti-Rabinowitz condition, such a sequence is not necessarily bounded. The existence of a *bounded* Palais-Smale sequence is actually not a trivial problem and the rest of the section is devoted to this issue. The arguments we use derive from the ones of [2].

**Lemma 12** For all  $t \in (t_*, 1)$  one has  $c = \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} J_\lambda(u^t)$ .

**Proof.** Letting  $t \in (t_*, 1)$  and  $c_t := \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} J_\lambda(u^t)$ , we show that  $c \leq c_t$  and  $c_t \leq c$ . For any  $\delta > 0$ , fix  $\gamma_1 \in \Gamma$  such that  $\max_{u \in \gamma_1([0,1])} J_\lambda(u^t) \leq c_t + \delta$  and set

$$\bar{\gamma}_1(s) := \begin{cases} \gamma_1(2s)^t & \text{if } 0 \leq s \leq 1/2 \\ 2(1-s)\bar{u}^t + 2(s-1/2)\bar{u} & \text{if } 1/2 \leq s \leq 1. \end{cases}$$

Then  $\bar{\gamma}_1 \in C([0,1]; H_{\lambda,s})$ ,  $\bar{\gamma}_1(0) = 0$  and  $\bar{\gamma}_1(1) = \bar{u}$ , i.e.,  $\bar{\gamma}_1 \in \Gamma$ . Moreover  $\bar{u}^t \in B_{\varepsilon_*}(\bar{u})$  implies  $\bar{\gamma}_1(s) \in B_{\varepsilon_*}(\bar{u})$  and thus  $J_\lambda(\bar{\gamma}_1(s)) < 0$  for all  $s \in [1/2, 1]$ . Hence we get

$$\begin{aligned} c &\leq \max_{u \in \bar{\gamma}_1([0,1])} J_\lambda(u) = \max_{s \in [0,1]} J_\lambda(\bar{\gamma}_1(s)) = \max_{s \in [0,1/2]} J_\lambda(\bar{\gamma}_1(s)) \\ &= \max_{s \in [0,1/2]} J_\lambda(\gamma_1(2s)^t) = \max_{s \in [0,1]} J_\lambda(\gamma_1(s)^t) = \max_{u \in \gamma_1([0,1])} J_\lambda(u^t) \\ &\leq c_t + \delta \end{aligned}$$

which yields  $c \leq c_t$  since  $\delta$  is arbitrary. Note that this also implies  $c_t > 0$ . Conversely, for any  $\delta > 0$  we fix  $\gamma_2 \in \Gamma$  such that  $\max_{u \in \gamma_2([0,1])} J_\lambda(u) \leq c + \delta$  and set

$$\bar{\gamma}_2(s) := \begin{cases} \gamma_2(2s)^{1/t} & \text{if } 0 \leq s \leq 1/2 \\ 2(1-s)\bar{u}^{1/t} + 2(s-1/2)\bar{u} & \text{if } 1/2 \leq s \leq 1. \end{cases}$$

Then  $\bar{\gamma}_2 \in C([0,1]; H_{\lambda,s})$ ,  $\bar{\gamma}_2(0) = 0$  and  $\bar{\gamma}_2(1) = \bar{u}$ , i.e.,  $\bar{\gamma}_2 \in \Gamma$ . Moreover, if  $s \in [1/2, 1]$ , one has  $\bar{\gamma}_2(s)^t = (2(1-s)\bar{u}^{1/t} + 2(s-1/2)\bar{u})^t = 2(1-s)\bar{u} + 2(s-1/2)\bar{u}^t$ , so that  $\bar{u}^t \in B_{\varepsilon_*}(\bar{u})$  implies  $\bar{\gamma}_2(s)^t \in B_{\varepsilon_*}(\bar{u})$  and so  $J_\lambda(\bar{\gamma}_2(s)^t) < 0$ . Hence we get

$$\begin{aligned} c_t &\leq \max_{u \in \bar{\gamma}_2([0,1])} J_\lambda(u^t) = \max_{s \in [0,1]} J_\lambda(\bar{\gamma}_2(s)^t) = \max_{s \in [0,1/2]} J_\lambda(\bar{\gamma}_2(s)^t) \\ &= \max_{s \in [0,1/2]} J_\lambda((\gamma_2(2s)^{1/t})^t) = \max_{s \in [0,1/2]} J_\lambda(\gamma_2(2s)) = \max_{s \in [0,1]} J_\lambda(\gamma_2(s)) \\ &= \max_{u \in \gamma_2([0,1])} J_\lambda(u) \leq c + \delta \end{aligned}$$

and the conclusion ensues from the arbitrariness of  $\delta$ . ■

Hereafter, by Lemma 12, we assume that to any  $t \in (t_*, 1)$  there corresponds a path  $\gamma_t \in \Gamma$  such that

$$\max_{u \in \gamma_t([0,1])} J_\lambda(u^t) \leq c + 1 - t^N, \quad (28)$$

by which we define the set

$$\Lambda_t := \{u \in \gamma_t([0,1]) : J_\lambda(u) \geq c - (1 - t^N)\}.$$

Note that  $\max_{u \in \gamma_t([0,1])} J_\lambda(u) \geq c$  implies  $\Lambda_t \neq \emptyset$  (indeed, as  $1 - t^N > 0$  and  $J_\lambda$  is continuous,  $\Lambda_t$  even contains a continuous piece of  $\gamma_t([0,1])$ ).



**Lemma 13** For every  $t \in (t_*, 1)$  and  $u \in \Lambda_t$  one has  $\|u\|_0^2 \leq (c+2)N/t_*^{N-2}$ .

**Proof.** Fix any  $t \in (t_*, 1)$  and  $u \in \Lambda_t$ . By (28) and the definition of  $\Lambda_t$  one has  $J_\lambda(u^t) - J_\lambda(u) \leq 2(1 - t^N)$ . On the other hand, from (22) and (23) it follows that

$$\begin{aligned} J_\lambda(u^t) - J_\lambda(u) &= \frac{1}{2} \left( \|u^t\|_0^2 - \|u\|_0^2 \right) - \mathcal{F}(u^t) + \mathcal{F}(u) \\ &= \frac{1}{2} \left( \|u^t\|_0^2 - \frac{1}{t^{N-2}} \|u^t\|_0^2 \right) - \mathcal{F}(u^t) + \frac{1}{t^N} \mathcal{F}(u^t) \\ &= \frac{1}{2} \left( 1 - \frac{1}{t^{N-2}} \right) \|u^t\|_0^2 - \left( 1 - \frac{1}{t^N} \right) \mathcal{F}(u^t) \\ &= -\frac{1-t^{N-2}}{2t^{N-2}} \|u^t\|_0^2 + \frac{1-t^N}{t^N} \mathcal{F}(u^t) \\ &= \frac{1-t^N}{t^N} \left( -\frac{1}{2} \frac{t^2-t^N}{1-t^N} \|u^t\|_0^2 + \mathcal{F}(u^t) \right) \\ &\geq \frac{1-t^N}{t^N} \left( -\frac{1}{2^*} \|u^t\|_0^2 + \mathcal{F}(u^t) \right), \end{aligned}$$

where for the final inequality we have taken into account that the mapping  $t \mapsto -(t^2 - t^N) / (1 - t^N)$  is decreasing for  $t > 0$  and tends to  $-2/2^*$  as  $t \rightarrow 1$ . So we obtain  $-(1/2^*) \|u^t\|_0^2 + \mathcal{F}(u^t) \leq 2t^N$ , whence

$$J_\lambda(u^t) = \frac{1}{2} \|u^t\|_0^2 - \mathcal{F}(u^t) \geq \frac{1}{2} \|u^t\|_0^2 - \frac{1}{2^*} \|u^t\|_0^2 - 2t^N = \frac{1}{N} \|u^t\|_0^2 - 2t^N$$

and therefore

$$\|u\|_0^2 = \frac{1}{t^{N-2}} \|u^t\|_0^2 \leq \frac{N}{t^{N-2}} (J_\lambda(u^t) + 2t^N) \leq \frac{N}{t^{N-2}} (c+1+t^N) \leq \frac{N(c+2)}{t_*^{N-2}},$$

where (28) has been used again in order to estimate  $J_\lambda(u^t) \leq c+1-t^N$ . ■

The aim of the section will be accomplished in Proposition 15 (and Corollary 16), where we take advantage of the following well known deformation lemma (see [35, Lemma 2.3], here written for the space  $H_{\lambda,s}$ , our functional  $J_\lambda$  and its mountain-pass level  $c$ ).

**Lemma 14** Let  $\mathcal{S} \subset H_{\lambda,s}$  and  $\varepsilon, \delta > 0$  be such that  $\|J'_\lambda(u)\|_{H'_{\lambda,s}} \geq 8\varepsilon/\delta$  for all  $u \in \mathcal{S}_{2\delta}$  satisfying  $|J_\lambda(u) - c| \leq 2\varepsilon$ , where

$$\mathcal{S}_{2\delta} := \left\{ v \in H_{\lambda,s} : \inf_{h \in \mathcal{S}} \|v - h\|_\lambda \leq 2\delta \right\}.$$

Then there exists  $\eta \in C([0, 1] \times H_{\lambda,s}; H_{\lambda,s})$  such that

- $\eta(\tau, u) = u$  provided that  $\tau = 0$  or  $|J_\lambda(u) - c| > 2\varepsilon$  or  $u \notin \mathcal{S}_{2\delta}$

- $J_\lambda(\eta(1, u)) \leq c - \varepsilon$  provided that  $J_\lambda(u) \leq c + \varepsilon$  and  $u \in \mathcal{S}$
- $\eta(\tau, \cdot)$  is an homeomorphism of  $H_{\lambda, s}$  for every  $\tau \in [0, 1]$
- $J_\lambda(\eta(\cdot, u))$  is nonincreasing for every  $u \in H_{\lambda, s}$ .

**Proposition 15** *There exists a Palais-Smale sequence  $\{w_n\} \subset H_{\lambda, s}$  for  $J_\lambda$  at level  $c$  such that*

$$\sup_n \|w_n\|_0^2 \leq 1 + 2 \frac{(c+2)N}{t_*^{N-2}}. \quad (29)$$

**Proof.** Set  $c_* := 2N(c+2)/t_*^{N-2}$  for sake of brevity. First we observe that  $\lim_{t \rightarrow 1^-} \sup_{u \in \Lambda_t} |J_\lambda(u^t) - J_\lambda(u)| = 0$ . Indeed, for all  $t \in (t_*, 1)$  and  $u \in \Lambda_t$ , the definition of  $\Lambda_t$  and (23) yield  $J_\lambda(u^t) - J_\lambda(u) \leq 2(1-t^N) = o(1)_{t \rightarrow 1}$  and  $-t^N \mathcal{F}(u) = -\mathcal{F}(u^t) \leq J_\lambda(u^t) \leq c+1-t^N$  by inequality (28), whence, by (22), (23) and Lemma 13, we get

$$\begin{aligned} J_\lambda(u) - J_\lambda(u^t) &= \frac{1-t^{N-2}}{2} \|u\|_0^2 - (1-t^N) \mathcal{F}(u) \\ &\leq (1-t^{N-2}) \frac{c_*}{4} + (1-t^N) \frac{c+1-t^N}{t^N} = o(1)_{t \rightarrow 1}. \end{aligned}$$

Now, for every  $m \geq 1$ , define

$$U_m := \left\{ u \in H_{\lambda, s} : \|u\|_0^2 \leq c_* + \frac{1}{m}, \quad |J_\lambda(u) - c| \leq \frac{1}{m} \right\}$$

and choose  $t_m \in (t_*, 1)$  such that  $1-t_m^N \leq 1/32m$  and  $J_\lambda(u) \leq J_\lambda(u^{t_m}) + 1/32m$  for all  $u \in \Lambda_{t_m}$ . Then for every  $u \in \Lambda_{t_m}$  the inequality of Lemma 13 holds and one has (recall the definition of  $\Lambda_{t_m}$ )

$$J_\lambda(u) \geq c - (1-t_m^N) \geq c - \frac{1}{32m}$$

and (by (28), with  $t = t_m$ )

$$J_\lambda(u) \leq J_\lambda(u^{t_m}) + \frac{1}{32m} \leq c + (1-t_m^N) + \frac{1}{32m} \leq c + \frac{1}{16m}, \quad (30)$$

whence  $\Lambda_{t_m} \subseteq U_m$  and  $U_m$  is not empty. For sake of contradiction, assume that

$$\exists \bar{m} > \max \left\{ \frac{8}{c_*}, \frac{1}{8c} \right\} \quad \forall u \in U_{\bar{m}} \quad \|J'_\lambda(u)\|_{H'_{\lambda, s}} \geq \frac{1}{\sqrt{\bar{m}}} \quad (31)$$

and apply Lemma 14 with  $\mathcal{S} = \{h \in H_{\lambda, s} : \|h\|_0^2 \leq c_*/2\}$ ,  $\varepsilon = 1/16\bar{m}$  and  $\delta = 1/2\sqrt{\bar{m}}$  (so that  $8\varepsilon/\delta = 1/\sqrt{\bar{m}}$ ). Note that

$$\mathcal{S}_{2\delta} = \mathcal{S}_{1/\sqrt{\bar{m}}} = \left\{ v \in H_{\lambda, s} : \min_{h \in \mathcal{S}} \|v - h\|_\lambda \leq \frac{1}{\sqrt{\bar{m}}} \right\}$$

because  $\mathcal{S}$  is convex and closed in  $H_{\lambda,s}$ , and observe that if  $u \in \mathcal{S}_{1/\sqrt{\bar{m}}}$  satisfies  $|J_\lambda(u) - c| \leq 1/8\bar{m}$  then  $u \in U_{\bar{m}}$  (and thus the last inequality of (31) holds), because there exists  $h \in \mathcal{S}$  such that  $\|u - h\|_0 \leq \|u - h\|_\lambda \leq 1/\sqrt{\bar{m}}$  and thus

$$\|u\|_0 \leq \|h\|_0 + \frac{1}{\sqrt{\bar{m}}} \leq \sqrt{\frac{c_*}{2}} + \frac{1}{\sqrt{\bar{m}}} \leq \sqrt{c_* + \frac{1}{\bar{m}}},$$

where the assumption  $\bar{m} > 8/c_*$  has been used to derive the last inequality. So there exists a homeomorphism  $\Phi : H_{\lambda,s} \rightarrow H_{\lambda,s}$  (namely  $\Phi := \eta(1, \cdot)$  of Lemma 14) such that

- (i)  $\Phi(u) = u$  if  $|J_\lambda(u) - c| \geq c$  (recall that  $c > 1/8\bar{m} = 2\varepsilon$ )
- (ii)  $J_\lambda(\Phi(u)) \leq c - 1/16\bar{m}$  if  $\|u\|_0^2 \leq c_*/2$  and  $J_\lambda(u) \leq c + 1/16\bar{m}$
- (iii)  $J_\lambda(\Phi(u)) \leq J_\lambda(u)$  for every  $u \in H_{\lambda,s}$ ,

by which we define the path  $\gamma := \Phi \circ \gamma_{t_{\bar{m}}} \in C([0, 1]; H_{\lambda,s})$ . By (i) one has  $\gamma(0) = \Phi(\gamma_{t_{\bar{m}}}(0)) = \Phi(0) = 0$  and  $\gamma(1) = \Phi(\gamma_{t_{\bar{m}}}(1)) = \Phi(\bar{u}) = \bar{u}$ , since  $|J_\lambda(\bar{u}) - c| = |J_\lambda(\bar{u})| + c > c$  (recall that  $J_\lambda(\bar{u}) < 0$ ). Hence  $\gamma \in \Gamma$ . We finally deduce the contradiction which assures that the hypothesis (31) is false and thus concludes the proof. Let  $u_* \in \gamma_{t_{\bar{m}}}([0, 1])$  be such that  $J(\Phi(u_*)) = \max_{u \in \gamma_{t_{\bar{m}}}([0, 1])} J_\lambda(\Phi(u)) = \max_{v \in \gamma([0, 1])} J_\lambda(v)$ . On one hand, if  $u_* \in \gamma_{t_{\bar{m}}}([0, 1]) \setminus \Lambda_{t_{\bar{m}}}$  then  $J_\lambda(\Phi(u_*)) \leq J_\lambda(u_*) < c - (1 - t_{\bar{m}}^N)$  (by (iii) and the definition of  $\Lambda_{t_{\bar{m}}}$ ). On the other hand, if  $u_* \in \Lambda_{t_{\bar{m}}}$  then (30) holds (with  $m = \bar{m}$ ) and Lemma 13 gives  $\|u_*\|_0^2 \leq c_*/2$  (recall that  $t_{\bar{m}} \in (t_*, 1)$ ), whence  $J_\lambda(\Phi(u_*)) \leq c - 1/16\bar{m}$  by (ii). Therefore, in any case one obtains  $\max_{v \in \gamma([0, 1])} J_\lambda(v) < c$ , which contradicts the definition (27) of  $c$ . ■

**Corollary 16** *The sequence  $\{w_n\}$  of Proposition 15 is bounded in  $H_{\lambda,s}$ .*

**Proof.** If  $\lambda = 0$  the assertion is already proved by (29); so assume  $\lambda > 0$ . Since  $J_\lambda(w_n) \rightarrow c$  and  $(\mathbf{F}_\vee)$  implies  $F(w_n) \leq M(|w_n|^p + |w_n|^q)$  almost everywhere, there exists a constant  $C_1 > 0$  such that

$$C_1 \geq J_\lambda(w_n) \geq \frac{\lambda}{2} \int_{\mathbb{R}^N} w_n^2 dx - M \int_{\mathbb{R}^N} |w_n|^p dx - M \int_{\mathbb{R}^N} |w_n|^q dx \quad \text{for all } n.$$

Setting  $\bar{p} := (2^* - 2)/(2^* - p)$  and  $\bar{p}' := \bar{p}/(\bar{p} - 1) = (2^* - 2)/(p - 2)$ , from Hölder and Sobolev inequalities one infers that there exists a second constant  $C_2 > 0$  such that

$$\begin{aligned} \int_{\mathbb{R}^N} |w_n|^p dx &= \int_{\mathbb{R}^N} |w_n|^{\frac{2}{\bar{p}}} |w_n|^{\frac{2^*}{\bar{p}'}} dx \leq \left( \int_{\mathbb{R}^N} w_n^2 dx \right)^{1/\bar{p}} \left( \int_{\mathbb{R}^N} |w_n|^{2^*} dx \right)^{1/\bar{p}'} \\ &\leq C_2 \|w_n\|_{L^2(\mathbb{R}^N)}^{2/\bar{p}} \|w_n\|_0^{2^*/\bar{p}'}. \end{aligned}$$

whence, by (29), there exists  $C_3 > 0$  such that  $\|w_n\|_{L^p(\mathbb{R}^N)}^p \leq C_3 \|w_n\|_{L^2(\mathbb{R}^N)}^{2/\bar{p}}$ . Similarly, there exists  $C_4 > 0$  such that  $\|w_n\|_{L^p(\mathbb{R}^N)}^q \leq C_4 \|w_n\|_{L^2(\mathbb{R}^N)}^{2/\bar{q}}$ , where  $\bar{q} := (2^* - 2) / (2^* - q)$ . Therefore we get

$$C_1 \geq \frac{\lambda}{2} \|w_n\|_{L^2(\mathbb{R}^N)}^2 - MC_3 \|w_n\|_{L^2(\mathbb{R}^N)}^{2/\bar{p}} - MC_4 \|w_n\|_{L^2(\mathbb{R}^N)}^{2/\bar{q}} \quad \text{for all } n,$$

where  $2/\bar{p}, 2/\bar{q} < 2$  since  $p, q > 2$ . Hence no diverging subsequence is allowed for  $\{\|w_n\|_{L^2(\mathbb{R}^N)}\}$  and the proof is thus complete. ■

## 4 Proof of Theorem 4

This section is devoted to the proof of Theorem 4, which relies on the application of a version of the concentration-compactness principle due to Solimini [28]. Accordingly, in order to state his result, we preliminarily introduce a group of rescaling operators, of which we also remark some basic properties.

As usual, we assume  $N > k \geq 2$  and let  $\mu > 0$ ,  $\lambda \geq 0$ .

**Definition 17** Let  $t > 0$  and  $x \in \mathbb{R}^N$ . For any  $u \in L^p(\mathbb{R}^N)$  with  $1 < p < \infty$  we define

$$T_{t,x}u := t^{-(N-2)/2} u(t^{-1} \cdot + x).$$

Clearly  $T_{t,x}u \in L^p(\mathbb{R}^N)$  for all  $u \in L^p(\mathbb{R}^N)$  and in particular  $T_{t,x}u \in D^{1,2}(\mathbb{R}^N)$  if  $u \in D^{1,2}(\mathbb{R}^N)$ . Moreover, by direct computations, it is easy to see that the linear operator  $u \mapsto T_{t,x}u$  is an isometry of both  $L^{2^*}(\mathbb{R}^N)$  and  $D^{1,2}(\mathbb{R}^N)$ . Notice that

$$T_{t,x}^{-1} = T_{1/t, -tx} \quad \text{and} \quad T_{t_1, x_1} T_{t_2, x_2} = T_{t_1 t_2, x_1/t_2 + x_2}. \quad (32)$$

**Remark 18** For any  $\tilde{z} = (0, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$  and  $t > 0$ , direct computations easily show that the linear operators  $u \mapsto T_{t,\tilde{z}}u$  and  $u \mapsto T_{1,\tilde{z}}u$  are isometries of  $H_0$  and  $H_\lambda$  respectively. Moreover  $T_{t,\tilde{z}}u \in H_{0,s}$  if  $u \in H_{0,s}$  and  $T_{1,\tilde{z}}u \in H_{\lambda,s}$  if  $u \in H_{\lambda,s}$ .

The next proposition is proved in [4].

**Proposition 19** Let  $1 < p < \infty$  and assume that  $\{t_n\} \subset (0, +\infty)$  and  $\{x_n\} \subset \mathbb{R}^N$  are such that  $t_n \rightarrow t \neq 0$  and  $x_n \rightarrow x$ . Then  $T_{t_n, x_n} u_n \rightharpoonup T_{t,x}u$  in  $L^p(\mathbb{R}^N)$  if  $u_n \rightharpoonup u$  in  $L^p(\mathbb{R}^N)$ .

**Corollary 20** Let  $\{t_n\} \subset (0, +\infty)$  and  $\{\tilde{z}_n\} \subset \{0\} \times \mathbb{R}^{N-k} \subset \mathbb{R}^N$  be such that  $t_n \rightarrow t \neq 0$  and  $\tilde{z}_n \rightarrow \tilde{z}$ . Then  $T_{t_n, \tilde{z}_n} u_n \rightharpoonup T_{t,\tilde{z}}u$  in  $H_{0,s}$  (up to a subsequence) if  $u_n \rightharpoonup u$  in  $H_{0,s}$ .

**Proof.** From the boundedness of  $\{u_n\}$ , by Remark 18 we deduce that also  $\{T_{t_n, \tilde{z}_n} u_n\}$  is bounded in  $H_{0,s}$ . Hence (up to a subsequence) it weakly converges

in  $H_{0,s}$  and  $L^{2^*}(\mathbb{R}^N)$ . On the other hand  $T_{t_n, \tilde{z}_n} u_n \rightharpoonup T_{t, \tilde{z}} u$  in  $L^{2^*}(\mathbb{R}^N)$ , because  $u_n \rightharpoonup u$  in  $L^{2^*}(\mathbb{R}^N)$  and Proposition 19 applies. ■

We are here in position to recall the above mentioned result of Solimini [28], which is the following.

**Theorem 21** *If  $\{v_n\} \subset D^{1,2}(\mathbb{R}^N)$  is bounded, then, up to a subsequence, either  $v_n \rightarrow 0$  in  $L^{2^*}(\mathbb{R}^N)$  or there exist  $\{t_n\} \subset (0, +\infty)$  and  $\{x_n\} \subset \mathbb{R}^N$  such that  $T_{t_n, x_n} v_n \rightharpoonup v$  in  $L^{2^*}(\mathbb{R}^N)$  and  $v \neq 0$ .*

Let us now turn to the proof of Theorem 4, which will be divided in several lemmas. Accordingly, we hereafter assume that all the hypotheses of the theorem are satisfied.

The starting point is the Palais-Smale sequence  $\{w_n\} \subset H_{\lambda,s}$  provided by Proposition 15, which, we recall, is bounded in  $H_{\lambda,s}$  (see Corollary 16) and satisfies  $J_\lambda(w_n) \rightarrow c > 0$  and  $J'_\lambda(w_n) \rightarrow 0$  in  $H'_{\lambda,s}$ .

As  $\{w_n\}$  is bounded in  $D^{1,2}(\mathbb{R}^N)$ , it must satisfy one of the alternatives allowed by Theorem 21. The following lemma shows that the first one cannot occur.

**Lemma 22** *The sequence  $\{w_n\}$  does not converge to 0 in  $L^{2^*}(\mathbb{R}^N)$ .*

**Proof.** Note that  $J'_\lambda(w_n) w_n \rightarrow 0$  since  $\{w_n\}$  is bounded in  $H_{\lambda,s}$  and, for sake of contradiction, assume that  $w_n \rightarrow 0$  in  $L^{2^*}(\mathbb{R}^N)$ . If  $\lambda = 0$  one can use  $(\mathbf{f}_*)$  and  $(\mathbf{F}_*)$  to readily deduce

$$\int_{\mathbb{R}^N} |f(w_n) w_n| dx + \int_{\mathbb{R}^N} |F(w_n)| dx \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (33)$$

which, by (20)-(21), yields the contradiction

$$\begin{aligned} J_\lambda(w_n) &= \frac{1}{2} \|w_n\|_\lambda^2 - \int_{\mathbb{R}^N} F(w_n) dx \\ &= \frac{1}{2} J'_\lambda(w_n) w_n + \frac{1}{2} \int_{\mathbb{R}^N} f(w_n) w_n dx - \int_{\mathbb{R}^N} F(w_n) dx = o(1)_{n \rightarrow \infty}. \end{aligned}$$

If  $\lambda > 0$ , then  $\{w_n\}$  is bounded in  $L^2(\mathbb{R}^N)$  so that  $w_n \rightarrow 0$  in  $L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$  by interpolation (recall that  $p, q \in (2, 2^*)$ ). Hence  $(\mathbf{f}_\vee)$  and  $(\mathbf{F}_\vee)$  imply (33) again and the same contradiction as before ensues. ■

**Corollary 23** *There exist  $\{t_n\} \subset (0, +\infty)$ ,  $\{x_n\} \subset \mathbb{R}^N$  and  $w \in L^{2^*}(\mathbb{R}^N)$ ,  $w \neq 0$ , such that (up to a subsequence)  $T_{t_n, x_n} w_n \rightharpoonup w$  in  $L^{2^*}(\mathbb{R}^N)$ .*

**Proof.** Apply Theorem 21 and use Lemma 22. ■

Now we can easily exploit the  $z$ -translation invariance of  $J_\lambda$  to improve the result of Corollary 23. To this end, we set  $x_n =: (y_n, z_n)$ ,  $\tilde{y}_n =: (y_n, 0)$  and  $\tilde{z}_n =: (0, z_n)$ , so that  $x_n = \tilde{y}_n + \tilde{z}_n$ , and define

$$u_n := T_{1, \tilde{z}_n} w_n.$$

**Lemma 24** *The sequence  $\{u_n\}$  is bounded in  $H_{\lambda,s}$  and satisfies  $J'_\lambda(u_n) \rightarrow 0$  in  $H'_{\lambda,s}$  and  $T_{t_n, \tilde{y}_n} u_n \rightharpoonup w$  in  $L^{2^*}(\mathbb{R}^N)$ .*

**Proof.** The boundedness of  $\{u_n\}$  follows from the one of  $\{w_n\}$ , since the operators  $T_{1, \tilde{z}_n}$  are isometries of  $H_{\lambda,s}$ . Moreover, by (21) and easy computations, one gets  $J'_\lambda(u_n)h = J'_\lambda(w_n)T_{1, -\tilde{z}_n}h$  for all  $h \in H_{\lambda,s}$ , so that  $\|J'_\lambda(u_n)\|_{H'_{\lambda,s}} = \|J'(w_n)\|_{H'_{\lambda,s}}$  because also  $T_{1, -\tilde{z}_n}$  are isometries of  $H_{\lambda,s}$ . Finally, recalling (32), we conclude  $T_{t_n, \tilde{y}_n} u_n = T_{t_n, \tilde{y}_n} T_{1, \tilde{z}_n} w_n = T_{t_n, x_n} w_n \rightharpoonup w$  in  $L^{2^*}(\mathbb{R}^N)$ . ■

By Lemmas 8 and 24, the sequence  $\{u_n\}$  weakly converges in  $H_{\lambda,s}$  to some critical point  $u \in H_{\lambda,s}$  of  $J_\lambda$ . The proof of Theorem 4 is thus accomplished if we show that  $u \neq 0$ , which is the aim of the next lemmas. The removal of translations from the rescalings  $T_{t_n, \tilde{y}_n}$  is the first step in that direction and it is the topic of the following lemma.

Hereafter we denote  $T_t := T_{t,0}$  for any  $t > 0$ .

**Lemma 25** *There exists  $v \in H_{0,s}$ ,  $v \neq 0$ , such that (up to a subsequence)  $T_{t_n} u_n \rightharpoonup v$  in  $H_{0,s}$ .*

**Proof.** Set  $v_n := T_{t_n} u_n$  for brevity and recall from Lemma 24 that  $T_{t_n, \tilde{y}_n} u_n \rightharpoonup w \neq 0$  in  $L^{2^*}(\mathbb{R}^N)$ . From Remark 18 we get  $v_n \in H_{0,s}$  and  $\|v_n\|_0 = \|u_n\|_0$ , so that (up to a subsequence) we can assume  $v_n \rightharpoonup v$  in  $H_{0,s}$ . If  $v \neq 0$  the proof is complete. So, for sake of contradiction, assume  $v_n \rightharpoonup 0$  in  $H_{0,s}$  (and thus in  $L^{2^*}(\mathbb{R}^N)$ ). First, we deduce that

$$\lim_{n \rightarrow \infty} |t_n \tilde{y}_n| = +\infty. \quad (34)$$

Otherwise, up to a subsequence,  $t_n \tilde{y}_n \rightarrow \tilde{y}_0 \in \mathbb{R}^k \times \{0\}$  and  $T_{1, -t_n \tilde{y}_n} T_{t_n, \tilde{y}_n} u_n \rightharpoonup T_{1, -\tilde{y}_0} w$  in  $L^{2^*}(\mathbb{R}^N)$  by Proposition 19. But, since  $T_{1, -t_n \tilde{y}_n} T_{t_n, \tilde{y}_n} = T_{t_n}$ , this means  $v_n \rightharpoonup T_{1, -\tilde{y}_0} w \neq 0$  in  $L^{2^*}(\mathbb{R}^N)$ , which is a contradiction. Now we observe that  $w \neq 0$  implies that there exist  $\delta > 0$  and  $A \subseteq \mathbb{R}^N$  with  $|A| \neq 0$  such that either  $w > \delta$  or  $w < -\delta$  almost everywhere in  $A$ . Then, fixing  $r > 0$  such that  $|B_r \cap A| > 0$ , by weak convergence we obtain

$$\left| \int_{\mathbb{R}^N} T_{t_n, \tilde{y}_n} u_n \chi_{B_r \cap A} dx \right| \rightarrow \left| \int_{\mathbb{R}^N} w \chi_{B_r \cap A} dx \right| \geq \delta |B_r \cap A| > 0. \quad (35)$$

On the other hand,  $T_{t_n, \tilde{y}_n} u_n = T_{t_n, \tilde{y}_n} T_{t_n}^{-1} v_n = T_{1, t_n \tilde{y}_n} v_n$  and hence

$$\begin{aligned} \left| \int_{\mathbb{R}^N} T_{t_n, \tilde{y}_n} u_n \chi_{B_r \cap A} dx \right| &\leq \int_{B_r} |T_{t_n, \tilde{y}_n} u_n| dx = \int_{B_r(t_n \tilde{y}_n)} |v_n| dx \\ &\leq C \left( \int_{B_r(t_n \tilde{y}_n)} |v_n|^{2^*} dx \right)^{1/2^*} \end{aligned} \quad (36)$$

for some constant  $C > 0$  which only depends on  $r$ . From (35) and (36) it follows that

$$\liminf_{n \rightarrow \infty} \int_{B_r(t_n \tilde{y}_n)} |v_n|^{2^*} dx > 0$$

and hence, up to a subsequence, we can assume

$$\inf_n \int_{B_r(t_n \tilde{y}_n)} |v_n|^{2^*} dx > \varepsilon_0 \quad \text{for some } \varepsilon_0 > 0. \quad (37)$$

This will yield a contradiction. Indeed, using (34), it is easy to see that for every  $l \in \mathbb{N}$ ,  $l \geq 2$ , there exists  $n_l \in \mathbb{N}$  such that for any  $n > n_l$  one can find  $R_1, \dots, R_l \in O(k)$  satisfying the condition

$$i \neq j \Rightarrow B_r(t_n(R_i y_n, 0)) \cap B_r(t_n(R_j y_n, 0)) = \emptyset$$

(see [4, Proposition 22] for a detailed proof). As a consequence, using (37) and the fact that  $v_n \in H_{0,s}$ , we get

$$\int_{\mathbb{R}^N} |v_n|^{2^*} dx \geq \sum_{i=1}^l \int_{B_r(t_n(R_i y_n, 0))} |v_n|^{2^*} dx = \sum_{i=1}^l \int_{B_r(t_n \tilde{y}_n)} |v_n|^{2^*} dx > l\varepsilon_0$$

for every natural numbers  $l \geq 2$  and  $n > n_l$ . This finally implies

$$\int_{\mathbb{R}^N} |v_n|^{2^*} dx \rightarrow +\infty$$

which is a contradiction, since  $\|v_n\|_{L^{2^*}(\mathbb{R}^N)} = \|T_{t_n} u_n\|_{L^{2^*}(\mathbb{R}^N)} = \|u_n\|_{L^{2^*}(\mathbb{R}^N)}$  and  $\{u_n\}$  is bounded in  $L^{2^*}(\mathbb{R}^N)$ . ■

According to Lemma 25 and in order to apply Corollary 20 with a view to concluding that  $\{u_n\}$  has a nonzero weak limit in  $H_{0,s}$  (and thus in  $H_{\lambda,s}$ ), we need to check that the dilation parameters  $\{t_n\}$  are bounded and bounded away from zero. This is the content of the remaining lemmas.

**Lemma 26** *If  $\lambda > 0$  then  $\inf_n t_n > 0$ .*

**Proof.** Recall from Lemma 25 that  $T_{t_n} u_n \rightharpoonup v \neq 0$  in  $H_{0,s} \hookrightarrow L_{loc}^2(\mathbb{R}^N)$  and fix  $\phi \in C_c^\infty(\mathbb{R}^N)$  and  $r > 0$  such that  $\text{supp } \phi \subset B_r$  and  $\int_{\mathbb{R}^N} v \phi dx \neq 0$ . Then  $T_{t_n} u_n \rightharpoonup v$  in  $L^2(B_r)$  and

$$\int_{\mathbb{R}^N} (T_{t_n} u_n) \phi dx = \int_{B_r} (T_{t_n} u_n) \phi dx \rightarrow \int_{B_r} v \phi dx = \int_{\mathbb{R}^N} v \phi dx \neq 0.$$

On the other hand,  $\{u_n\}$  is bounded in  $L^2(\mathbb{R}^N)$  and we get

$$\begin{aligned} \left| \int_{\mathbb{R}^N} (T_{t_n} u_n) \phi dx \right| &\leq \|\phi\|_{L^2(\mathbb{R}^N)} \left( \int_{\mathbb{R}^N} |T_{t_n} u_n|^2 dx \right)^{1/2} \\ &= \|\phi\|_{L^2(\mathbb{R}^N)} \left( \int_{\mathbb{R}^N} \left| t_n^{-(N-2)/2} u_n(t_n^{-1}x) \right|^2 dx \right)^{1/2} \\ &= \|\phi\|_{L^2(\mathbb{R}^N)} \left( t_n^2 \int_{\mathbb{R}^N} u_n^2 dx \right)^{1/2} \\ &\leq t_n \|\phi\|_{L^2(\mathbb{R}^N)} \left( \sup_n \|u_n\|_{L^2(\mathbb{R}^N)} \right) \rightarrow 0. \end{aligned}$$

As a conclusion, no vanishing subsequence is allowed for  $\{t_n\}$  and the claim follows. ■

**Lemma 27** *If  $\lambda > 0$  then  $\sup_n t_n < +\infty$ .*

**Proof.** Recall from Lemma 25 that  $T_{t_n} u_n \rightharpoonup v$  in  $H_{0,s}$  and  $\|v\|_0 > 0$ . Then, by Proposition 6, there exists  $\tilde{v} \in H_{\lambda,s}$  such that  $(v | \tilde{v})_0 > 0$ , so that

$$(u_n | T_{1/t_n} \tilde{v})_0 = (u_n | T_{t_n}^{-1} \tilde{v})_0 = (T_{t_n} u_n | \tilde{v})_0 \rightarrow (v | \tilde{v})_0 > 0 \quad (38)$$

by (32) and Remark 18. For sake of contradiction, up to a subsequence we now assume  $t_n \rightarrow +\infty$ . Then  $T_{t_n}^{-1} \tilde{v} = T_{1/t_n} \tilde{v} \in H_{0,s}$  with  $\|T_{1/t_n} \tilde{v}\|_0 = \|\tilde{v}\|_0$  and

$$\int_{\mathbb{R}^N} |T_{1/t_n} \tilde{v}|^2 dx = \int_{\mathbb{R}^N} |t_n^{(N-2)/2} \tilde{v}(t_n x)|^2 dx = t_n^{-2} \int_{\mathbb{R}^N} |\tilde{v}|^2 dx \rightarrow 0,$$

which implies that  $\{T_{1/t_n} \tilde{v}\}$  is bounded in  $H_{\lambda,s}$  and  $L^{2^*}(\mathbb{R}^N)$ , and thus it converges to zero in  $L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$  by interpolation (recall that  $p, q \in (2, 2^*)$ ). Hence (recall Lemma 24)  $J'_\lambda(u_n) T_{1/t_n} \tilde{v} \rightarrow 0$  and

$$\begin{aligned} \int_{\mathbb{R}^N} |u_n T_{1/t_n} \tilde{v}| dx &\leq \|u_n\|_{L^2(\mathbb{R}^N)} \|T_{1/t_n} \tilde{v}\|_{L^2(\mathbb{R}^N)} \\ &\leq \left( \sup_n \|u_n\|_{L^2(\mathbb{R}^N)} \right) \|T_{1/t_n} \tilde{v}\|_{L^2(\mathbb{R}^N)} \rightarrow 0. \end{aligned}$$

Moreover, by  $(\mathbf{f}_v)$  and the boundedness of  $\{u_n\}$  in  $L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ , there exists a constant  $C > 0$  such that

$$\begin{aligned} \int_{\mathbb{R}^N} |f(u_n) T_{1/t_n} \tilde{v}| dx &\leq m \int_{\mathbb{R}^N} (|u_n|^{p-1} + |u_n|^{q-1}) |T_{1/t_n} \tilde{v}| dx \\ &\leq m \left( \int_{\mathbb{R}^N} |u_n|^p dx \right)^{(p-1)/p} \left( \int_{\mathbb{R}^N} |T_{1/t_n} \tilde{v}|^p dx \right)^{1/p} + \\ &\quad + m \left( \int_{\mathbb{R}^N} |u_n|^q dx \right)^{(q-1)/q} \left( \int_{\mathbb{R}^N} |T_{1/t_n} \tilde{v}|^q dx \right)^{1/q} \\ &\leq C \left( \|T_{1/t_n} \tilde{v}\|_{L^p(\mathbb{R}^N)} + \|T_{1/t_n} \tilde{v}\|_{L^q(\mathbb{R}^N)} \right) \rightarrow 0. \end{aligned}$$

Therefore by (21) we obtain

$$(u_n | T_{1/t_n} \tilde{v})_0 = J'_\lambda(u_n) T_{1/t_n} \tilde{v} - \lambda \int_{\mathbb{R}^N} u_n T_{1/t_n} \tilde{v} dx + \int_{\mathbb{R}^N} f(u_n) T_{1/t_n} \tilde{v} dx \rightarrow 0$$

which contradicts (38). ■

**Lemma 28** *If  $\lambda = 0$  then  $0 < \inf_n t_n \leq \sup_n t_n < +\infty$ .*



**Proof.** Recall from Lemma 25 that  $T_{t_n} u_n \rightharpoonup v \neq 0$  in  $H_{0,s}$ . Then, by Proposition 6, there exists  $\tilde{v} \in H_{0,s} \cap L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$  such that  $(v | \tilde{v})_0 > 0$ , whence

$$(u_n | T_{1/t_n} \tilde{v})_0 = (u_n | T_{t_n}^{-1} \tilde{v})_0 = (T_{t_n} u_n | \tilde{v})_0 \rightarrow (v | \tilde{v})_0 > 0 \quad (39)$$

by (32) and Remark 18. Now observe that, setting  $p' := p/(p-1)$  and  $q' := q/(q-1)$ ,  $p < 2^* < q$  implies  $(p-1)q' < 2^* < (q-1)p'$ , so that condition  $(\mathbf{f}_\wedge)$  gives

$$\max \left\{ |f(t)|^{p'}, |f(t)|^{q'} \right\} \leq (m^{p'} + m^{q'}) |t|^{2^*} \quad \text{for all } t \in \mathbb{R}.$$

Hence, as  $\{u_n\}$  is bounded in  $L^{2^*}(\mathbb{R}^N)$ ,  $\{f(u_n)\}$  is bounded in  $L^{p'}(\mathbb{R}^N) \cap L^{q'}(\mathbb{R}^N)$  and thus there exist  $C_1, C_2 > 0$  such that

$$\begin{aligned} \int_{\mathbb{R}^N} |f(u_n) T_{1/t_n} \tilde{v}| dx &\leq \left( \int_{\mathbb{R}^N} |f(u_n)|^{p'} dx \right)^{1/p'} \left( \int_{\mathbb{R}^N} |T_{1/t_n} \tilde{v}|^p dx \right)^{1/p} \\ &\leq C_1 \left( t_n^{\frac{N-2}{2} p - N} \int_{\mathbb{R}^N} |\tilde{v}|^p dx \right)^{1/p} \\ &= C_1 t_n^{\frac{N-2}{2p}(p-2^*)} \|\tilde{v}\|_{L^p(\mathbb{R}^N)} \end{aligned}$$

and, similarly,

$$\int_{\mathbb{R}^N} |f(u_n) T_{1/t_n} \tilde{v}| dx \leq C_2 t_n^{\frac{N-2}{2q}(q-2^*)} \|\tilde{v}\|_{L^q(\mathbb{R}^N)}.$$

Since  $p < 2^* < q$ , this implies  $\int_{\mathbb{R}^N} |f(u_n) T_{1/t_n} \tilde{v}| dx \rightarrow 0$  (up to a subsequence) either if  $t_n \rightarrow 0$  or if  $t_n \rightarrow +\infty$  (up to a subsequence) and therefore one deduces

$$(u_n | T_{1/t_n} \tilde{v})_0 = J'_0(u_n) T_{1/t_n} \tilde{v} + \int_{\mathbb{R}^N} f(u_n) T_{1/t_n} \tilde{v} dx \rightarrow 0$$

since  $J'_0(u_n) \rightarrow 0$  in  $H'_{0,s}$  (Lemma 24, with  $\lambda = 0$ ) and  $T_{1/t_n} \tilde{v}$  is bounded in  $H_{0,s}$  because  $\|T_{1/t_n} \tilde{v}\|_0 = \|\tilde{v}\|_0$ . So a contradiction ensues with (39) if the assertion of the lemma is false. ■

We are now able to easily conclude the proof of Theorem 4.

**Proof of Theorem 4.** By the last Lemmas 26-28, up to a subsequence we can assume  $t_n \rightarrow t \neq 0$ . Thus, from  $T_{t_n} u_n \rightharpoonup v \neq 0$  in  $H_{0,s}$  (Lemma 25) we deduce  $u_n \rightharpoonup T_t^{-1} v \neq 0$  in  $H_{0,s}$  (up to a subsequence) by Corollary 20. Therefore, recalling from Lemma 24 that  $\{u_n\}$  is bounded in  $H_{\lambda,s} \hookrightarrow H_{0,s}$  for every  $\lambda$ , one infers that  $u_n \rightharpoonup T_t^{-1} v$  in  $H_{\lambda,s}$  also for  $\lambda > 0$ . Finally, since  $J'_\lambda(u_n) \rightarrow 0$  in  $H'_{\lambda,s}$  (see Lemma 24 again), Lemma 8 assures that  $T_t^{-1} v \in H_{\lambda,s}$  is a (nonzero) critical point for  $J_\lambda$ . The conclusion then follows from Proposition 7. ■

## 5 Proof of Theorem 1

In this section we give the proof of Theorem 1, which follows from Theorem 4 together with an extendibility argument aimed at removing of the singularity of  $\nabla\theta$  on the plane  $y = 0$ , where  $\theta$  is the angular coordinate given by (8).

Let  $W : \mathbb{C} \rightarrow \mathbb{R}$  satisfy (2) and assume all the hypotheses of Theorem 1. Let  $k_0 \neq 0$  and  $\omega_0 \in (\Omega_0, \Omega]$ , with  $\omega_0 \in (\Omega_0, \Omega)$  if  $2 < q \leq 6$  in hypothesis (ii). Set

$$G(s) := \frac{1}{2}\Omega^2 s^2 - V(s) \quad \text{for all } s \in \mathbb{R}.$$

In order to apply Theorem 4 with  $N = 3$ ,  $k = 2$ ,  $\mu = k_0^2$ ,  $\lambda = \Omega^2 - \omega_0^2$  and  $g = G'$ , one readily checks that  $(\mathbf{g}_0)$  and  $(\mathbf{g}_1)$  are satisfied. We just observe that, if  $\lambda > 0$ , definition (6) implies the existence of  $\omega \in (\Omega_0, \omega_0)$  and  $s_0 > 0$  such that

$$G(s_0) - \frac{1}{2}\lambda s_0^2 = \frac{1}{2}\omega_0^2 s_0^2 - V(s_0) > \frac{1}{2}\omega^2 s_0^2 - V(s_0) > 0.$$

Moreover,  $V \geq 0$  implies

$$\limsup_{s \rightarrow +\infty} \frac{g(s)}{s^{p-1}} < +\infty \quad \text{for every } p > 2. \quad (40)$$

So, if  $g(s) \geq 0$  for  $s > 0$  large, then (40) assures that  $(\mathbf{g}_3)$  holds. Otherwise, if there exists a sequence  $\{s_n\}$  such that  $s_n \rightarrow +\infty$  and  $g(s_n) < 0$ , it is not difficult to deduce  $(\mathbf{g}_2)$  from  $(\mathbf{g}_0)$  and  $(\mathbf{g}_1)$ . Therefore Theorem 4 provides equation (9) with a nonzero nonnegative solution  $u \in H_s$  in the following weak sense:

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla h \, dx + k_0^2 \int_{\mathbb{R}^3} \frac{uh}{|y|^2} \, dx + \int_{\mathbb{R}^3} V'(u) h \, dx = \omega_0^2 \int_{\mathbb{R}^3} uh \, dx \quad \text{for all } h \in H. \quad (41)$$

Note that, either if  $(\mathbf{g}_2)$  holds or if  $(\mathbf{g}_3)$  holds, one has that

$$V'(u) = \Omega^2 u - g(u) \in L_{loc}^1(\mathbb{R}^3). \quad (42)$$

Moreover, according to definitions 20 and 19, we have

$$I_{\Omega^2 - \omega_0^2}(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left[ |\nabla u|^2 + \frac{k_0^2}{|y|^2} u^2 + (\Omega^2 - \omega_0^2) u^2 \right] dx - \int_{\mathbb{R}^3} G(u) \, dx,$$

so that (10) becomes

$$\mathcal{E} \left( u(x) e^{i(k_0\theta(x) - \omega_0 t)} \right) = I_{\Omega^2 - \omega_0^2}(u) + \omega_0^2 \int_{\mathbb{R}^3} u^2 dx < \infty.$$

Now we set

$$\psi_0(x) := u(x) e^{ik_0\vartheta(x)} \quad \text{for all } x = (y, z) \in \mathcal{O} := (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}.$$

Notice that  $\nabla\psi_0 = e^{ik_0\vartheta}(\nabla u + ik_0 u \nabla\vartheta)$  implies  $|\nabla\psi_0|^2 = |\nabla u|^2 + k_0^2 |y|^{-2} u^2 \in L^1(\mathbb{R}^3)$ , so that  $\psi_0 \in H^1(\mathbb{R}^3)$ . In order to conclude the proof of Theorem 1, we need the following two lemmas.

**Lemma 29** *The mapping  $\psi_0$  satisfies*

$$-\Delta\psi_0 + W'(\psi_0) = \omega_0^2\psi_0 \quad (43)$$

*in the distributional sense on  $\mathcal{O}$ .*

**Proof.** Since  $\theta \in C^\infty(\mathcal{O}; \mathbb{R}/2\pi\mathbb{Z})$ , the claim of the lemma is equivalent to

$$\int_{\mathcal{O}} \nabla\psi_0 \cdot \nabla(e^{-ik_0\vartheta}\varphi) dx + \int_{\mathcal{O}} W'(\psi_0) e^{-ik_0\vartheta}\varphi dx = \omega_0^2 \int_{\mathcal{O}} \psi_0 e^{-ik_0\vartheta}\varphi dx \quad (44)$$

for all  $\varphi \in C_c^\infty(\mathcal{O}; \mathbb{C})$ , where  $W'(\psi_0) = V'(u) e^{ik_0\vartheta}$  by (2). Writing  $\varphi = \varphi_1 + i\varphi_2$  with  $\varphi_1, \varphi_2 \in C_c^\infty(\mathcal{O}; \mathbb{R})$ , we readily get

$$\int_{\mathcal{O}} W'(\psi_0) e^{-ik_0\vartheta}\varphi dx = \int_{\mathcal{O}} V'(u) \varphi dx = \int_{\mathbb{R}^3} V'(u) \varphi_1 dx + i \int_{\mathbb{R}^3} V'(u) \varphi_2 dx$$

and

$$\int_{\mathcal{O}} \psi_0 e^{-ik_0\vartheta}\varphi dx = \int_{\mathcal{O}} u\varphi dx = \int_{\mathbb{R}^3} u\varphi_1 dx + i \int_{\mathbb{R}^3} u\varphi_2 dx.$$

On the other hand, denoting  $\xi \cdot \eta = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3$  for any  $\xi, \eta \in \mathbb{C}^3$ , one has

$$\begin{aligned} & \int_{\mathcal{O}} \nabla\psi_0 \cdot \nabla(e^{-ik_0\vartheta}\varphi) dx \\ &= \int_{\mathcal{O}} (\nabla u + ik_0 u \nabla\vartheta) \cdot (\nabla\varphi - ik_0 \varphi \nabla\vartheta) dx \\ &= \int_{\mathcal{O}} \left( \nabla u \cdot \nabla\varphi + k_0^2 u \varphi |\nabla\vartheta|^2 \right) dx + ik_0 \int_{\mathcal{O}} u \nabla\vartheta \cdot \nabla\varphi dx \\ &= \int_{\mathcal{O}} \left( \nabla u \cdot \nabla\varphi + k_0^2 \frac{u\varphi}{|y|^2} \right) dx - ik_0 \int_{\mathcal{O}} \operatorname{div}(u \nabla\vartheta) \varphi dx \\ &= \int_{\mathcal{O}} \left( \nabla u \cdot \nabla\varphi + k_0^2 \frac{u\varphi}{|y|^2} \right) dx \\ &= \int_{\mathbb{R}^3} \left( \nabla u \cdot \nabla\varphi_1 + k_0^2 \frac{u\varphi_1}{|y|^2} \right) dx + i \int_{\mathbb{R}^3} \left( \nabla u \cdot \nabla\varphi_2 + k_0^2 \frac{u\varphi_2}{|y|^2} \right) dx \end{aligned}$$

where we have taken into account that  $\nabla u \cdot \nabla\vartheta = 0$  and  $\operatorname{div}(u \nabla\vartheta) = \nabla u \cdot \nabla\vartheta + u \Delta\vartheta = 0$ . Hence, observing that  $C_c^\infty(\mathcal{O}; \mathbb{R}) \subset H$ , one concludes that (44) holds thanks to (41). ■

**Lemma 30** *The mapping  $\psi_0$  satisfies (43) in the distributional sense on  $\mathbb{R}^3$ .*

**Proof.** Let  $\varphi \in C_c^\infty(\mathbb{R}^3; \mathbb{C})$  and take  $\{\eta_n\} \subset C^\infty(\mathbb{R}^3; \mathbb{R})$  such that  $\eta_n \rightarrow 1$  almost everywhere in  $\mathbb{R}^3$  and

- $0 \leq \eta_n \leq 1$ ,  $\eta_n(y, z) = 0$  for  $|y| \leq 1/n$  and  $\eta_n(y, z) = 1$  for  $|y| \geq 2/n$
- $|\nabla \eta_n| \leq (\text{const.}) n$  on  $\mathbb{R}^3$ .

Clearly  $\eta_n \varphi \in C_c^\infty(\mathcal{O}; \mathbb{C})$  and  $|\nabla \eta_n(y, z)| = 0$  for  $|y| \leq 1/n$  or  $|y| \geq 2/n$ . Then Lemma 29 gives

$$\int_{\mathbb{R}^3} \varphi \nabla \psi_0 \cdot \nabla \eta_n dx + \int_{\mathbb{R}^3} \eta_n \nabla \psi_0 \cdot \nabla \varphi dx + \int_{\mathbb{R}^3} W'(\psi_0) \varphi \eta_n dx = \int_{\mathbb{R}^3} \psi_0 \varphi \eta_n dx. \quad (45)$$

Setting  $A_n := \{(y, z) \in \text{supp } \varphi : 1/n < |y| < 2/n\}$ , we have  $|A_n| \leq (\text{const.})/n^2$  and

$$\begin{aligned} \left| \int_{\mathbb{R}^3} \varphi \nabla \psi_0 \cdot \nabla \eta_n dx \right| &\leq (\text{const.}) \|\varphi\|_{L^\infty(\mathbb{R}^3)} n \int_{A_n} |\nabla \psi_0| dx \\ &\leq (\text{const.}) \left( \int_{A_n} |\nabla \psi_0|^2 dx \right)^{1/2} \rightarrow 0. \end{aligned}$$

Passing to the limit in (45) and using the Lebesgue's dominated convergence theorem for the other terms (recall that  $|W'(\psi_0)| = |V'(u)| \in L_{loc}^1(\mathbb{R}^3)$  by (2) and (42)), the claim follows. ■

**Proof of Theorem 1.** Set  $\psi(t, x) := \psi_0(x) e^{-i\omega_0 t}$  for all  $x \in \mathcal{O}$  and  $t \in \mathbb{R}$ . Since Lemma 30, together with standard elliptic regularity arguments (see for example [20]), yields that  $\psi_0$  defines a classical solution to (43) on  $\mathbb{R}^3$ , a straightforward substitution proves that  $\psi$  is actually a classical solution of (1) on  $\mathbb{R} \times \mathbb{R}^3$ . ■

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